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An Approach to Modelling and Simulation of Uncertain Dynamical Systems

EYKE HÜLLERMEIER
Heinz Nixdorf Institut, University of Paderborn
eyke@hni.uni-paderborn.de

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Coping with uncertainty in dynamical systems has recently received some attention in artificial intelligence (AI), particularly in the fields of qualitative and model-based reasoning. In this paper, we propose an approach to modelling and simulation of uncertain dynamics which is based on the following ideas: We consider (linguistic) descriptions of uncertain functional relationships characterizing the behavior of some dynamical system. Based on a certain interpretation of such rule-based models, we derive a fuzzy function $\widetilde{F}$. It will be shown that all (reasonable) fuzzy functions can be approximated to any degree of accuracy in this way. The function $\widetilde{F}$ is then used as the "fuzzy" right hand side of a set of differential equations, which leads us to consider *fuzzy initial value problems*. We are going to propose an interpretation of such problems. Moreover, several aspects of simulation methods for characterizing the set of all system behaviors compatible with this interpretation will be discussed.

Keywords: linguistic modelling, fuzzy functions, approximation, dynamical systems, fuzzy differential equations, reachable sets.

1. Introduction

Knowledge about dynamical systems modelled by (ordinary) differential equations (ODEs) is often incomplete or vague. For example, parameter values, functional relationships, or initial conditions may not be known precisely. In this situation, well-known methods for solving initial value problems analytically or numerically can only be used for finding selected system behaviors, e.g., by fixing unknown parameters to some plausible values. But it is generally not possible to characterize the whole set of system behaviors compatible with our partial knowledge this way. However, it is just this kind of information which is often important in applications of knowledge-based systems, such as, e.g., model-based monitoring and diagnosis [16].

A special kind of uncertainty is vagueness in natural language. Models based on natural language can be seen as a vague formalization of mental models, which is in many cases more adequate than precise mathematical models: "When traditional simulation models of social phenomena are formulated, causal relations are represented as precise mathematical functions. Such is the case even when the modeler has only a vague idea about their nature, a condition which is most often true ... . To avoid the artificial step of translating vague ideas with an inappropriate exactitude, the modeler should instead be allowed to formulate his model in natural language." [46] Linguistic models have been applied successfully in, e.g., fuzzy control. Moreover, natural language simulation has also received attention in AI [5]. Of course, natural language models cannot be used directly for simulating the behavior of dynamical systems: "Humans speak and write in natural language; however, there must be a translation process [mapping qualitative to quantitative models] if this knowledge is to be useful to simulation." [18] This leads us to consider the problem of mapping natural language descriptions of functional relationships, which we suppose to be given as if-then rules, to a mathematical representation in form of a fuzzy function $\widetilde{F}$. In Section 3, such rule-based models will be shown to be capable of approximating all "reasonable" fuzzy functions to any degree of accuracy.

In this paper, we are going to propose an approach to modelling and simulation of uncertain dynamical systems. The general idea is to characterize the set of all system behaviors compatible with (the mathematical interpretation of) a natural language description of the system. We pass from (ordinary) initial value problems to *fuzzy initial value problems* by replacing the right hand side $\mathsf{f} : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ of an ODE system by some fuzzy function $\widetilde{F} : [0,T] \times \mathbb{R}^n \to \mathbb{F}(\mathbb{R}^n)$, where $\mathbb{F}(\mathbb{R}^n)$ is the set of all fuzzy subsets of $\mathbb{R}^n$. Likewise, initial system states $\mathsf{x}_0 \in \mathbb{R}^n$ are replaced by *initial fuzzy sets* $\widetilde{X}_0 \in \mathbb{F}(\mathbb{R}^n)$. Of course, mathematical models in form of fuzzy initial value problems no longer describe a unique system behavior (such as ODE systems satisfying a uniqueness condition.) The following questions have to be answered in connection with this kind of mathematical models: What is the interpretation of the model from a semantical point of view? What is meant by a solution to a fuzzy initial value problem? How can the set of possible system behaviors be characterized?

The paper is organized as follows: Methods and results related to modelling of uncertain functional relationships are presented in Section 2 and Section 3. In Section 4, we are going to discuss the questions just mentioned in connection with fuzzy initial value problems. Some aspects of numerical methods for fuzzy initial value problems as well as an example of it are presented in Section 5. Section 6 gives a brief overview of some related methods proposed in the literature.

2. Linguistic Modelling

Fuzzy systems are widely used for linguistic modelling of functional relationships. A well-known example is fuzzy control: The incomplete and vague knowledge about a control function is formulated as a set of linguistic rules by a human expert. Fuzzy inference is used to transform the set of fuzzy rules into a mathematical function $\mathsf{g}(\cdot)$ which serves as an approximation of the true but unknown function $\mathsf{f}(\cdot)$. Fuzzy systems such as those used in fuzzy control generate real-valued functions $\mathsf{g} : X \to Y$ from an input space $X \subset \mathbb{R}^m$ to an output space $Y \subset \mathbb{R}^p$ by performing the three steps (1) *fuzzification of input values*, (2) *fuzzy inference*, (3) *defuzzification of output values*. The defuzzification procedure assigns a real vector $\mathsf{a} \in \mathbb{R}^p$ to the fuzzy output $\widetilde{A}$ of the inference mechanism. This can be interpreted as follows: First, the fuzzy system generates a fuzzy function $\widetilde{G} : X \to \mathbb{F}(Y)$ from the input space X to the space $\mathbb{F}(Y)$ of all fuzzy subsets of the set Y. Then, the defuzzification procedure realizes a *selection* $\mathsf{g} : X \to Y$ satisfying $\mathsf{g}(\mathsf{x}) \in \widetilde{G}(\mathsf{x})$ for all $\mathsf{x} \in X$. This is meaningful if such a "crisp" function is needed as, e.g., in fuzzy control. Of course, some information is lost in the last step. This can be avoided by taking $\widetilde{G}$ itself instead of a selection $\mathsf{g}(\cdot)$ as the output of the fuzzy system in which case the fuzzy function $\widetilde{G}$ is interpreted as a quantification of a vague idea of a functional relationship $\mathsf{f}(\cdot)$.

Consider a fuzzy system S given by n rules of the form

$$R_i : \quad \text{If} \quad \mathsf{x} \in \widetilde{A}_i \quad \text{then} \quad \mathsf{y} \in \widetilde{B}_i \ , \tag{1}$$

where $\widetilde{A}_i \in \mathbb{F}(\mathbb{R}^m)$, $\widetilde{B}_i \in \mathbb{F}(\mathbb{R}^p)$ are fuzzy sets representing some linguistic variables like `small` or `medium`. The mapping which is realized by this fuzzy system highly depends on the choice of

- the membership functions $\mu_{\widetilde{A}_i}$ and $\mu_{\widetilde{B}_i}$,
- the fuzzy inference which specifies a fuzzy output for each input $\mathsf{x} \in \mathbb{R}^m$,
- the defuzzification method which maps a fuzzy set $\widetilde{B} \in \mathbb{F}(\mathbb{R}^p)$ to a "crisp" value $\mathsf{y} \in \mathbb{R}^p$.

Since we are interested in the fuzzy output of the system, only the first two aspects are relevant here. A defuzzification of the output is not considered.

2.1. *Rule-based Fuzzy Systems*

Denote by $\mathcal{E}^p$ the set of fuzzy sets $\widetilde{A} \in \mathbb{F}(\mathbb{R}^p)$ which are normal, upper semicontinuous, fuzzy convex, and compactly supported. The α-level set of a fuzzy set $\widetilde{A}$ is defined as $[\mu_{\widetilde{A}}]_\alpha := \{\mathsf{x} \in X \mid \mu_{\widetilde{A}}(\mathsf{x}) \geq \alpha\}$. The set $[\mu_{\widetilde{A}}]_0$ is defined as the closure of $\operatorname{supp}(\widetilde{A}) := \{\mathsf{x} \in X \mid \mu_{\widetilde{A}}(\mathsf{x}) > 0\}$. We consider m input variables $x_k \in \mathbb{R}$ and an output $\mathsf{y} \in \mathbb{R}^p$ with corresponding domains $X_k \subset \mathbb{R}$ and $Y \subset \mathbb{R}^p$, respectively, where X_k is compact. Let $\mathsf{x} := (x_1, \ldots, x_m)$ and $X := X_1 \times \ldots \times X_m$. Thus, the fuzzy system S consisting of n rules (1) realizes a fuzzy function $\widetilde{G} : X \to \mathbb{F}(Y)$. We suppose $\widetilde{B}_i \in \mathcal{E}^p$ $(i = 1, \ldots, n)$. Furthermore, the membership functions $\mu_{\widetilde{A}_i}$ are supposed to satisfy the following consistency condition:

$$\forall \mathsf{x} \in X \ \exists j \in \{1, \ldots, n\} \ : \ \mu_{\widetilde{A}_j}(\mathsf{x}) > 0 \ . \tag{2}$$

Usually, (1) takes the form

$$R_i : \quad \text{If} \quad x_1 \in \widetilde{A}_{i1} \quad \text{and} \quad x_2 \in \widetilde{A}_{i2} \quad \text{and} \quad \ldots \quad \text{and} \quad x_m \in \widetilde{A}_{im} \quad \text{then} \quad \mathsf{y} \in \widetilde{B}_i \ ,$$

where the $\widetilde{A}_{ik} \in \mathbb{F}(\mathbb{R})$ are characterized by membership functions $\mu_{\widetilde{A}_{ik}} : X_k \to [0,1]$ for $k \in \{1,\ldots,m\}$. Then, $\mu_{\widetilde{A}_i}$ is a function of $\mu_{\widetilde{A}_{i1}},\ldots,\mu_{\widetilde{A}_{im}}$. For example, $\mu_{\widetilde{A}_i}(\mathsf{x}) := \mathbf{T}_{k=1}^{m}\mu_{\widetilde{A}_{ik}}(x_k)$ for some t-norm $\top$. Normally, the single output case ($p=1$) is considered in rule-based modelling [48]. This is justified by the following: For $\mathsf{y} = (y_1,\ldots,y_p)$ the conclusion ($\mathsf{y} \in \widetilde{B}$) is equivalent to

$$y_1 \in \widetilde{B}_1 \quad \text{and} \quad y_2 \in \widetilde{B}_2 \quad \text{and} \quad \ldots \quad \text{and} \quad y_p \in \widetilde{B}_p \quad ,$$

where $\widetilde{B}_j \in \mathbb{F}(\mathbb{R})$ $(j = 1,\ldots,p)$. That is, a rule $R:$ If ($\mathsf{x} \in \widetilde{A}$) then ($\mathsf{y} \in \widetilde{B}$) can be split into p rules

$$\begin{array}{lllll} R_1: & \text{If} & \mathsf{x} \in \widetilde{A} & \text{then} & y_1 \in \widetilde{B}_1 \\ R_2: & \text{If} & \mathsf{x} \in \widetilde{A} & \text{then} & y_2 \in \widetilde{B}_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ R_p: & \text{If} & \mathsf{x} \in \widetilde{A} & \text{then} & y_p \in \widetilde{B}_p \end{array} ,$$

which leads to p separate rule-based fuzzy systems. However, here the implicit assumption is made that the variables $y_1,\ldots,y_p$ are *noninteractive* [15], which means that $\widetilde{B} = \mathcal{P}_1(\widetilde{B})\times\ldots\times\mathcal{P}_p(\widetilde{B})$, where $\mathcal{P}_k(\widetilde{B})$ is the projection of the fuzzy relation $\widetilde{B}$ to Y_k, the domain of the variable y_k. Thus, noninteractivity of $y_1,\ldots,y_p$ or *separability* of $\widetilde{B}$ means nothing else than $\mu_{\widetilde{B}}(\mathsf{y}) = \min\{\mu_{\widetilde{B}_1}(y_1),\ldots,\mu_{\widetilde{B}_p}(y_p)\}$. However, this assumption is not always justified. Consider the vague description of a position in the plane as an example. A membership value of a point (u,v) is determined best as a function of the Euklidean distance from a reference point (u_0,v_0), which means that the variables u and v are interactive. Thus, it is advantageous to take the more general multi-output case into account.

2.2. *Fuzzy Inference*

We will define the output $\widetilde{Y} = \widetilde{G}(\mathsf{x})$ of a fuzzy system S for an input $\mathsf{x} \in X$ as the weighted average of the reference output values $\widetilde{B}_i$ $(i = 1,\ldots,n)$.

Definition 1 (fuzzy basic function (FBF)) *For a fuzzy system S described above n fuzzy basic functions* [45,48] *b_j $(j=1,\ldots,n)$ are defined by means of*

$$b_j : X \to [0,1]\,,\, \mathsf{x} \mapsto \frac{\mu_{\widetilde{A}_j}(\mathsf{x})}{\sum_{i=1}^{n}\mu_{\widetilde{A}_i}(\mathsf{x})} \ .$$

From (2) follows that the FBFs are well defined. Obviously, $\sum_{i=1}^{n} b_i(\mathsf{x}) = 1$ for all $\mathsf{x} \in X$. FBFs can be seen as nonlinear combinations of radial basis functions [31].

The output $\widetilde{Y} = \widetilde{G}(\mathsf{x})$ of the fuzzy system S is defined by

$$\widetilde{G} : X \to \mathbb{F}(Y)\,,\, \mathsf{x} \mapsto \sum_{i=1}^{n} b_i(\mathsf{x}) \cdot \widetilde{B}_i \ .$$

That is, the weight of a rule R_i $(i = 1,\ldots,n)$ in the fuzzy inference process is defined by the relation of the values of the membership functions $\mu_{\widetilde{A}_i}$ at $\mathsf{x} \in X$

which can be interpreted as the truth degrees of the corresponding premises. The conclusion $(\mathsf{y} \in \widetilde{Y})$ is a weighted average of the n predefined conclusions $(\mathsf{y} \in \widetilde{B}_i)$.

3. Universal Approximation Property

In this section, we will show the rule-based fuzzy systems defined in Section 2 as capable of approximating any continuous fuzzy function with normal, upper semicontinuous, fuzzy-convex, and compactly supported values on a compact set to arbitrary degree. The result combines research efforts from different directions. On the one side, it extends theoretical results concerning the approximation capability of fuzzy systems for real-valued functions [19]. Wang [44,45] shows that additive fuzzy systems with Gaussian membership functions and centroid defuzzification can approximate any real continuous function on a compact set to arbitrary degree. Similar results were obtained by Buckley [9], Kosko [34], and Zeng and Singh [48,49]. Our approach also uses additive fuzzy systems. However, the crisp output is replaced by a fuzzy one.

On the other side, this result is closely related with the analysis of set-valued functions. The approximation of set-valued functions was first investigated by Vitale [43] and later by Artstein [3] and Keimel and Roth [30]. Particularly, we make use of results of Diamond and Ramer [14] who generalized the results obtained in [43] and [30] for Bernstein approximation and Korovkin systems for fuzzy functions. The interpolation of fuzzy data was investigated by Lowen [36] and Kaleva [28]. Approximation problems have also been discussed in [2] and [21]. Fuzzy interpolation and approximation problems in connection with rules are treated in [1,33,41] and [39,40]. However, the approximation power of fuzzy systems in not addressed in these works.

Definition 2 (Property $p(\varepsilon,\delta)$) *Let $\varepsilon,\delta > 0$. Let $d(.,.)$ be a metric on X. A fuzzy system S as described above is said to have property $p(\varepsilon,\delta)$ if the following holds true:*

$$\forall\, i \in \{1,\dots,n\}\ \exists\, \mathsf{z}_i \in \operatorname{supp}(\widetilde{A}_i)\ \forall\, \mathsf{x} \in X\ :\ d(\mathsf{x},\mathsf{z}_i) \geq \delta \Rightarrow b_i(\mathsf{x}) \leq \varepsilon\ . \tag{3}$$

For $\mathsf{x} \in X$, $\varepsilon > 0$, let $I_1(\mathsf{x},\varepsilon) := \{i \in \{1,\dots,n\} \,|\, b_i(\mathsf{x}) > \varepsilon\}$. Furthermore, let $I_2(\mathsf{x},\varepsilon) := \{1,\dots,n\} \setminus I_1(\mathsf{x},\varepsilon)$.

Loosely spoken, property (3) states that the influence of each rule in the fuzzy system S becomes small outside a certain area. Observe that $I_1(\mathsf{x},\varepsilon) \neq \emptyset$ for $\varepsilon < 1/n$.

The definition of property $p(\varepsilon,\delta)$ is based on the FBFs b_i. Thus, it cannot be infered directly from the membership functions $\mu_{\widetilde{A}_i}$ if $p(\varepsilon,\delta)$ is satisfied. However, the following estimation is possible. The value

$$\lambda := \min_{\mathsf{x}\in X}\ \max_{0\leq i\leq n}\ \mu_{\widetilde{A}_i}(\mathsf{x}) \tag{4}$$

can be interpreted as a consistency measure. The consistency assumption (2) is equivalent to $\lambda > 0$. Obviously, the fuzzy system satisfies property $p(\varepsilon,\delta)$ if

$$\forall\, i \in \{1,\dots,n\}\ \exists\, \mathsf{z}_i \in \operatorname{supp}(\widetilde{A}_i)\ \forall\, \mathsf{x} \in X\ :\ d(\mathsf{x},\mathsf{z}) \geq \delta \Rightarrow \mu_{\widetilde{A}_i}(\mathsf{x}) \leq \lambda\varepsilon\ .$$

In mathematical approximation theory, some regularity assumptions such as continuity of the corresponding functions always are made. Then, it is well known that approximation accuracy at a point $x \in X$ increases if local information about the function $f : X \to Y$ in a neighbourhood $U(x)$ of x is added. Consider piecewise linear approximation as an example. The more values of $f(\cdot)$ are known, the more accurate is the approximation. The same idea is reflected by the property $p(\varepsilon, \delta)$. The smaller ε and δ, the more fuzzy rules are needed in order to satisfy a certain consistency degree, the more local information is used for approximating a function $\widetilde{F}$. Consider the case of a 1-dimensional input space $X = [x_0, x_1]$. The membership functions in the premises of the fuzzy rules form a fuzzy partition [32] of X. As the partition gets finer, the approximation power of the fuzzy system improves. Consider the case where $\varepsilon = 0$. For a system with property $p(0, \delta)$ it follows that $\mathrm{diam}(\mathrm{supp}(\widetilde{A}_i)) \leq 2\,\delta$, where $\mathrm{diam}([\alpha, \beta]) = \beta - \alpha$. Thus, in order to satisfy the consistency condition, $n \geq \lceil (x_1 - x_0)/(2\,\delta) \rceil$ is necessary.

The properties of the function $\widetilde{G}$ depend more or less obviously on the membership functions $\mu_{\widetilde{A}_i}$ $(i = 1, \ldots, n)$. If all these functions are continuous, so is $\widetilde{G}$. Of course, $\widetilde{G}(\mathsf{x}) \in \mathcal{E}^p$ for all $\mathsf{x} \in X$.

A complete metric space $(\mathcal{E}^p, \tilde{d}_H)$ is defined on a compact set X by endowing $\mathcal{E}^p$ with the metric

$$\tilde{d}_H(\widetilde{A}, \widetilde{B}) := \sup_{\alpha \in [0,1]} d_H([\mu_{\widetilde{A}}]_\alpha, [\mu_{\widetilde{B}}]_\alpha) \ .$$

Here, d_H is the Hausdorff metric on the space of nonempty and compact subsets of $\mathbb{R}^p$:

$$d_H(X, Y) := \max\{\beta(X, Y), \beta(Y, X)\} \ ,$$

where $\beta(X, Y) := \sup\{\rho(\mathsf{x}, Y) \,|\, \mathsf{x} \in X\}$ and $\rho(\mathsf{x}, Y) = \min_{\mathsf{y} \in Y} |\mathsf{x} - \mathsf{y}|$ with a norm $|.|$ on $\mathbb{R}^n$. Subsequently, by continuity of fuzzy functions and convergence of fuzzy sets we mean continuity and convergence with respect to $\tilde{d}_H$.

We associate with each $\widetilde{A} \in \mathcal{E}^p$ its support function $S_{\widetilde{A}}$ defined by

$$S_{\widetilde{A}} : [0, 1] \times S^p \to \mathbb{R} \ , \ (\alpha, \mathsf{y}) \mapsto \max\{\langle \mathsf{y}, \mathsf{x} \rangle \,|\, \mathsf{x} \in [\mu_{\widetilde{A}}]_\alpha\} \ ,$$

where $\langle ., . \rangle$ is the inner product and $S^p := \{\mathsf{x} \in \mathbb{R}^p \,|\, |\mathsf{x}|_2 = 1\}$ is the unit sphere in $\mathbb{R}^p$. It is shown in [13] that the map $\pi(\widetilde{A}) = S_{\widetilde{A}}$ is a linear homeomorphic embedding from $\mathcal{E}^p$ into the Banach space $\mathcal{H} = L_\infty(S^p, BV([0, 1])) \cap L_\infty([0, 1], C(S^p))$ with the norm

$$|S_{\widetilde{A}}| = \sup_{\mathsf{y} \in S^p} V(S_{\widetilde{A}}(., \mathsf{y})) + \sup_{\alpha \in [0,1]} \sup_{\mathsf{y} \in S^p} |S_{\widetilde{A}}(\alpha, \mathsf{y})| \ ,$$

where $V(.)$ is the total variation, $BV([0, 1])$ is the space of functions having bounded variation on $[0, 1]$, and $L_\infty(U, V)$ is the space of bounded functions from a compact space U into a normed space V with norm $|f| = \sup_{\mathsf{u} \in U} |f(\mathsf{x})|_V$.

Consider a sequence of fuzzy systems S_q with n_q the number of rules in S_q. Let the function e_q be defined by

$$e_q : \mathbb{R} \to [0, 1] \ , \ \delta \mapsto \max_{i \in \{1, \ldots, n_q\}} \{b_i(\mathsf{x}) \,|\, \mathsf{x} \in X, |\mathsf{x} - \mathsf{z}_i| \geq \delta\} \ .$$

When proving Theorem 1 below, we face the following problem: For $\varepsilon, \delta > 0$ we have to find a fuzzy system S satisfying property $p(\varepsilon/n, \delta)$, with n the number of rules in S. The question arises if such a fuzzy system exists regardless of the values ε and δ. This question is equivalent to the claim that a sequence of systems S_q exists for a compact set $X \subset \mathbb{R}^m$ with $e_q(\delta) = o(1/n_q)$ for fixed $\delta > 0$. That is, $\lim_{n\to\infty} n_q\, e_q(\delta) = 0$. Of course, this follows at once from the possibility of constructing a fuzzy system with property $p(0, \delta)$ based on triangular membership functions and product inference logic. However, we would like to demonstrate that such a sequence S_q can also be found in the "nontrivial" case where $b_i(\mathsf{x}) > 0$ on X $(i = 1, \dots, n)$.

Lemma 1 *Let $\delta > 0$, $X \subset \mathbb{R}^m$ with $X \subset [x_{10}, x_{11}] \times \dots \times [x_{m0}, x_{m1}]$. A sequence of fuzzy systems S_q with domain X exists such that $e_q(\delta) = o(1/n_q)$ with n_q the number of rules in S_q.*

Proof. We will prove this lemma by constructing explicitely a sequence S_q of fuzzy systems which has the desired property. Let $\delta > 0$ be fixed. For $\Delta > 0$ define $r_k := \lceil (x_{k1} - x_{k0})/\Delta \rceil$ and $d_k := (x_{k1} - x_{k0})/r_k \leq \Delta$, $k \in \{1, \dots, m\}$. Let $\mathcal{J} := \{\mathsf{j} = (j_1, \dots, j_m) \mid 0 \leq j_1 \leq r_1, \dots, 0 \leq j_m \leq r_m\}$ and $\mathsf{z}_\mathsf{j} := (x_{10} + j_1\, d_1, \dots, x_{m0} + j_m\, d_m)$ for $1 \leq k \leq m$, $0 \leq j_k \leq r_k$. Consider the fuzzy sets $\widetilde{A}_\mathsf{j}$ defined by

$$\mu_{\widetilde{A}_\mathsf{j}}(\mathsf{x}) := \exp\left(\frac{-4\ln(2)\, |\mathsf{x} - \mathsf{z}_\mathsf{j}|_2^2}{m\, \Delta^2} \right) .$$

The number of rules in S_q depends on Δ and is given by $n_q = r_1 \cdot r_2 \cdot \ldots \cdot r_k$. From the construction above follows that for each $\mathsf{x} \in X$ we find $\mathsf{j} \in \mathcal{J}$ such that $|\mathsf{x} - \mathsf{z}_\mathsf{j}|_2 \leq \sqrt{m}\, \Delta/2$. Thus, for the value λ in (4) we get $\lambda \geq 1/2$. Therefore,

$$e_q(\delta) \leq 2 \exp\left(\frac{-4\ln(2)\, \delta^2}{m\, \Delta^2} \right) = \exp(-O((1/\Delta)^2)) .$$

Thus, the result follows from $n_q = n_q(\Delta) = O((1/\Delta)^m)$ and $x^a = o(\exp(x^b))$ for $a, b \in \mathbb{N}$. $\square$.

Theorem 1 *Let $X \subset \mathbb{R}^m$ be compact. Let $\widetilde{F} : X \to \mathcal{E}^p$ be a continuous fuzzy function. Then, a sequence of fuzzy systems S_q exists, the associated fuzzy functions $\widetilde{G}_q$ of which satisfy $\lim_{q\to\infty} \widetilde{G}_q = \widetilde{F}$ uniformly in X. That is, $\tilde{d}_H(\widetilde{G}_q(\mathsf{x}), \widetilde{F}(\mathsf{x})) \to 0$ uniformly in X as $q \to \infty$.*

Proof. Continuity of $\widetilde{F}$ is equivalent to continuity of $S_{\widetilde{F}}$ [14]. Consider the set of values $\mathsf{z}_1, \dots, \mathsf{z}_n$ from (3) and let $\widetilde{B}_i := \widetilde{F}(\mathsf{z}_i)$ $(i = 1, \dots, n)$ in the fuzzy system S. For the corresponding fuzzy function $\widetilde{G}$, we have

$$S_{\widetilde{G}(\mathsf{x})} = \sum_{i=1}^{n} b_i(\mathsf{x}) S_{\widetilde{B}_i} .$$

Theorem 1 follows from $S_{\widetilde{G}(\mathsf{x})} \to S_{\widetilde{F}(\mathsf{x})}$ uniformly in X, which will now be shown. From $\sum_{i=1}^n b_i \equiv 1$ and $(\alpha + \beta)\widetilde{A} = \alpha\widetilde{A} + \beta\widetilde{A}$ for $\widetilde{A} \in \mathcal{E}^p$ follows that

$$\widetilde{F}(\mathsf{x}) = \sum_{i=1}^{n} b_i(\mathsf{x}) \cdot \widetilde{F}(\mathsf{x}) .$$

Since $S_{\tilde{F}}$ is continuous,

$$\forall\, \varepsilon > 0\, \exists\, \delta > 0 \,:\, |\mathsf{x}_1 - \mathsf{x}_2| \le \delta \Rightarrow |S_{\tilde{F}(\mathsf{x}_1)} - S_{\tilde{F}(\mathsf{x}_2)}| \le \varepsilon\ . \tag{5}$$

Furthermore, $M > 0$ exists such that $|S_{\tilde{F}(\mathsf{x})}| \le M$ on X. Now, let $\varepsilon > 0$ and choose $\delta > 0$ from (5) for $\varepsilon/2$. Consider the fuzzy function $\widetilde{G}$ associated with a fuzzy system S satisfying property $p(\varepsilon_1, \delta)$, where $\varepsilon_1 = \varepsilon/(4nM)$. The existence of such a system is guaranteed by Lemma 1. It follows that

$$\begin{aligned}
\left|S_{\tilde{G}(\mathsf{x})} - S_{\tilde{F}(\mathsf{x})}\right| &= \sum_{i=1}^{m} b_i(\mathsf{x}) \left|S_{\tilde{F}(\mathsf{z}_i)} - S_{\tilde{F}(\mathsf{x})}\right| \\
&= \sum_{i \in I_1(\mathsf{x},\varepsilon_1)} b_i(\mathsf{x}) \left|S_{\tilde{F}(\mathsf{z}_i)} - S_{\tilde{F}(\mathsf{x})}\right| + \sum_{i \in I_2(\mathsf{x},\varepsilon_1)} b_i(\mathsf{x}) \left|S_{\tilde{F}(\mathsf{z}_i)} - S_{\tilde{F}(\mathsf{x})}\right| \\
&\le \varepsilon/2 + 2nM\varepsilon_1 \le \varepsilon\ .
\end{aligned}$$

Since $\widetilde{F}$ is continuous on X, it is uniformly continuous. Thus, the estimates above are uniform in X. $\square$.

4. Simulation of Uncertain Dynamical Systems

In this section, we are going to propose a method for simulating dynamical systems modelled by differential equations with set-valued or “fuzzy” right hand side. The method is based on the theory of differential inclusions. First, we are going to consider *generalized initial value problems.* Here, the model of a dynamical system is given in the form of a differential inclusion and an initial system state restricted by some set $X_0 \subset \mathbb{R}^n$. Then, *fuzzy initial value problems* will be introduced as a further generalization: Set-valued functions are replaced by fuzzy functions and initial sets by fuzzy initial sets. Based on a probabilistic interpretation of this kind of model, we establish a certain relation between “set-valued” and “fuzzy” models.

4.1. *Generalized Initial Value Problems*

One possibility of modelling (bounded) uncertainty in a dynamical system is to replace functions and initial values in the problem

$$\dot{\mathsf{x}}(t) = \mathsf{f}_0(t, \mathsf{x}(t)), \qquad \mathsf{x}(t_0) = \mathsf{x}_0 \in \mathbb{R}^n \tag{6}$$

by set-valued functions and initial sets. This leads to the following (generalized) initial value problem:

$$\dot{\mathsf{x}}(t) \in F(t, \mathsf{x}(t)), \qquad \mathsf{x}(t_0) \in X_0 \subset \mathbb{R}^n\ , \tag{7}$$

where $F : [t_0, T] \times \mathbb{R}^n \to 2^{\mathbb{R}^n} \setminus \{\emptyset\}$ is a set-valued function, X_0 is compact and convex. A solution $\mathsf{x}(\cdot)$ of (7) is understood to be an absolutely continuous function $\mathsf{x} : [t_0, T] \to \mathbb{R}^n$ which satisfies (7) almost everywhere. The function F is taken to be set-valued in order to represent the (bounded) uncertainty of the dynamical system:

For each system state $(t, \mathsf{x}) \in [t_0, T] \times \mathbb{R}^n$ the derivative is not known precisely, but an element of the set $F(t, \mathsf{x})$. Let

$$\mathcal{X} := \{\mathsf{x} : [t_0, T] \to \mathbb{R}^n \,|\, \mathsf{x}(\cdot) \text{ is a solution of (7)}\} \ . \tag{8}$$

Then, the *reachable set* $X(t)$ at time $t \in [t_0, T]$ is defined as

$$X(t) = X(t; t_0, X_0) := \{\mathsf{x}(t) \,|\, \mathsf{x}(\cdot) \in \mathcal{X}\} \ .$$

The function $X(t; t_0, X_0)$ satisfies a semigroup property [47], namely

$$X(t; \tau, X(\tau; t_0, X_0)) = X(t; t_0, X_0) \tag{9}$$

for $t_0 \leq \tau \leq t \leq T$. If we regard the whole set of real-valued solutions (8) as one set-valued trajectory, (9) can be interpreted as a generalized dynamical system.

The reachable set $X(t)$ is the set of all possible system states at time t. Knowledge about $X(t)$ is important for many applications. Thus, it seems meaningful to characterize the behavior of uncertain dynamical systems by means of reachable sets. Our objective is to find approximations of these sets for the system (7) using numerical methods. In order to define meaningful approximation procedures, it is necessary to know some properties of these sets. Furthermore, we need a theoretical basis which allows solving (7) numerically, i.e., a discrete approximation of (7). Before we turn to these aspects, however, we will consider a further generalization of (7).

4.2. *Fuzzy Initial Value Problems*

A reasonable generalization of "set-valued" modelling, which takes aspects of gradedness into account, is the replacement of sets by fuzzy sets, i.e., (7) becomes the fuzzy initial value problem

$$\dot{\mathsf{x}}(t) \in \widetilde{F}(t, \mathsf{x}(t)), \quad \mathsf{x}(0) \in \widetilde{X}_0 \tag{10}$$

on $J = [0, T]$ with a fuzzy function $\widetilde{F} : J \times \mathbb{R}^n \to \mathcal{E}^n$ and a fuzzy set $\widetilde{X}_0 \in \mathcal{E}^n$, where $\mathcal{E}^n$ is the set of normal, upper semicontinuous, fuzzy convex, and compactly supported fuzzy sets $\widetilde{X} \in \mathbb{F}(\mathbb{R}^n)$.

In order to define the meaning of a solution to (10), we consider this kind of modelling from a semantic point of view. Our interpretation is a probabilistic one. We assume the existence of a probability space $(\mathcal{C} \times D, \mathcal{A}, \mu)$ modelling the uncertainty concerning the unknown function f_0 and the unknown initial system state x_0 in (6). Here, $\mathcal{C}$ is a certain class of functions, and D is a set of possible initial system states. The tuple $(\mathsf{f}_0, \mathsf{x}_0)$ can be thought of as a fixed but unknown "parameter" of the initial value problem (6). In this case, the probability μ is interpreted (in a subjective sense) as a (conditional) distribution based on a body of knowledge, i.e., as a (subjective) quantification of a degree of confirmation concerning the "value" of $(\mathsf{f}_0, \mathsf{x}_0)$. For instance, the knowledge that $\mathsf{x}_0 = 0$ and $\mathsf{f}_0(t, \mathsf{x}) = \mathsf{x} + \alpha$ with unknown $\alpha \in [0, 1]$

can be modelled by means of $\mathcal{C} = \{\mathsf{f} : J \times \mathbb{R}^n \to \mathbb{R}^n, (t, \mathsf{x}) \mapsto \mathsf{x} + \alpha \,|\, \alpha \in [0,1]\}$, $D = \{0\}$, and μ the uniform measure on $\mathcal{C} \times D$.

The fuzzy right hand side in (10) is regarded as "weak" information about the probability μ: For certain values $\alpha \in [0,1]$, the set-valued $(1-\alpha)$-section $F_{1-\alpha}$ of $\widetilde{F}$ together with the $(1-\alpha)$-cut $[\mu_{\widetilde{X}_0}]_{1-\alpha}$ are associated with an α-confidence region $\mathcal{C}_\alpha \times D_\alpha \subset \mathcal{C} \times D$ for the true but unknown tuple $(\mathsf{f}_0, \mathsf{x}_0)$. More precisely, $\mathcal{C}_\alpha$ will be defined as the set of all functions $\mathsf{f} \in \mathcal{C}$ satisfying $\mathrm{graph}(\mathsf{f}) \subset \mathrm{graph}(F_{1-\alpha})$. Likewise, D_α is defined as $[\mu_{\widetilde{X}_0}]_{1-\alpha} \subset \mathbb{R}^n$. Thus, we obtain

$$\mathrm{Prob}\left(\mathrm{graph}(\mathsf{f}_0) \subset \mathrm{graph}(F_\alpha) \wedge \mathsf{x}_0 \in [\mu_{\widetilde{X}_0}]_\alpha\right) = 1 - \alpha \ .$$

It should be noted that an alternative interpretation of $(\mathsf{f}_0, \mathsf{x}_0)$ (in an objective sense) as a random variable modelled by the probability space $(\mathcal{C} \times D, \mathcal{A}, \mu)$ is just as well possible.

According to our interpretation, it is logical to consider the generalized initial value problems

$$\dot{\mathsf{x}}(t) \in F_\alpha(t, \mathsf{x}(t)), \quad \mathsf{x}(0) \in [\mu_{\widetilde{X}_0}]_\alpha \ , \tag{11}$$

where $F_\alpha(t, \mathsf{z}) := [\mu_{\widetilde{F}(t,\mathsf{z})}]_\alpha$ is the α-cut of the fuzzy set $\widetilde{F}(t, \mathsf{z})$. F_0 is defined pointwise as $F_0(t, \mathsf{x}) := \overline{\mathrm{supp}}(\widetilde{F}(t, \mathsf{x}))$. We call a function $\mathsf{x} : J \to \mathbb{R}^n$ an α-solution to (10) if it is absolutely continuous and satisfies (11) almost everywhere on J. The set of all α-solutions to (10) is denoted $\mathcal{X}_\alpha$, and the α-reachable sets $X_\alpha(t)$ are defined as

$$X_\alpha(t) := \{\mathsf{x}(t) \,|\, \mathsf{x}(\cdot) \in \mathcal{X}_\alpha\} \ .$$

Loosely spoken, $\mathcal{X}_\alpha$ is thought of as an $(1-\alpha)$-confidence region for the (unknown) solution* $\mathsf{x}_0 : J \to \mathbb{R}^n$ to (6) and defines the α-section of the fuzzy set $\widetilde{X}$ of solutions to (10). Likewise, $X_\alpha(t)$ is a confidence region for the value $\mathsf{x}_0(t) \in \mathbb{R}^n$. $X_\alpha(t)$ defines the α-cut of the fuzzy reachable set $\widetilde{X}(t)$.† The following results provide the formal basis for this interpretation. Firstly, it will be shown that the set $\mathcal{X}$ of solutions of a generalized initial value problem corresponds with the set of solutions associated with "ordinary" problems $\dot{\mathsf{x}} = \mathsf{f}(t, \mathsf{x})$, where f is a Carathéodory selection of F. Therefore, we will define $\mathcal{C}$ as the class of all functions $\mathsf{f}(t, \mathsf{x})$ measurable in t and continuous in x. Secondly, the existence of a probability measure compatible with the confidence regions associated with a fuzzy function $\widetilde{F}$ will be shown.

Proposition 1 *Let F be continuous, bounded with compact convex values. Consider a generalized initial value problem (7). Then $\mathcal{X} = \mathcal{X}'$, where $\mathcal{X}'$ is the set of all functions solving an initial value problem*

$$\dot{\mathsf{x}} = \mathsf{f}(t, \mathsf{x}) \ \textit{almost everywhere on } [0,T],\ \mathsf{x}(0) \in X_0$$

with $\mathsf{f} \in \mathcal{F}$, where

$$\mathcal{F} := \{\mathsf{g} : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n \,|\, \mathsf{g} \textit{ is a Carathéodory selection of } F\} \ . \tag{12}$$

*Actually, we should speak of the set of solutions, since (6) need not have a unique solution.

†The system $\{X_\alpha(t) \,|\, 0 < \alpha \leq 1\}$ completely determines $\widetilde{X}(t)$.

Proof. Evidently, $\mathcal{X}' \subset \mathcal{X}$ holds. Let $\mathsf{x}(\cdot) \in \mathcal{X}$. Since $\mathsf{x}(\cdot)$ is absolutely continuous, $\dot{\mathsf{x}}(\cdot)$ has the Lusin property, i.e., for all $\varepsilon > 0$ a closed set $J_\varepsilon \subset [0,T]$ exists such that $\mu([0,T] \setminus J_\varepsilon) \leq \varepsilon$ and $\dot{\mathsf{x}}(\cdot)|J_\varepsilon$ is continuous (μ is the Lebesgue measure). The set $[0,T]$ can therefore be written as $[0,T] = \bigcup_{m \geq 0} J_m$ with disjoint sets $J_m \subset [0,T]$ such that $\mu(J_0) = 0$, J_m is closed and $\dot{\mathsf{x}}(\cdot)|J_m$ is continuous for any $m \geq 1$. Define $G : [0,T] \times \mathbb{R}^n \to 2^{\mathbb{R}^n} \setminus \{\emptyset\}$ by means of

$$G(t, \mathsf{u}) = \begin{cases} \{\dot{\mathsf{x}}(t)\} & \text{if } t \in \bigcup_{m \geq 1} J_m, \mathsf{x}(t) = \mathsf{u} \\ F(t, \mathsf{u}) & \text{otherwise} \end{cases} .$$

$G|(J_m \times \mathbb{R}^n)$ is lsc for all $m \geq 1$. Thus, Michael's selection theorem (see [4]) guarantees the existence of a continuous selection g_m of $G|(J_m \times \mathbb{R}^n)$. Now, let

$$\mathsf{f}(t, \mathsf{u}) = \begin{cases} \{\mathsf{g}_m(t, \mathsf{u})\} & \text{if } t \in J_m \\ 0 & \text{if } t \in J_0 \end{cases} .$$

We have $\mathsf{f}(t, \mathsf{u}) \in F(t, \mathsf{u})$ for all $t \in \bigcup_{m \geq 1} J_m$ and $\mathsf{x} \in \mathbb{R}^n$, and $\mathsf{f}(t, \mathsf{x}(t)) = \dot{\mathsf{x}}(t)$ almost everywhere. Moreover, f is "almost continuous," i.e., for all $\varepsilon > 0$ a closed set $J_\varepsilon \subset [0,T]$ exists such that f is continuous on $J_\varepsilon \times \mathbb{R}^n$ and $\mu([0,T] \setminus J_\varepsilon) \leq \varepsilon$. Hence, f is Carathéodory. $\square$.

According to Proposition 1, we consider the set of all Carathéodory selections of the set-valued function F as possible candidates for the unknown right hand side f_0 in (6) when solving a generalized initial value problem. Denote by $\mathcal{C}$ the class of all functions $\mathsf{f} : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ which are measurable in the first argument and continuous in the second one. Consider a fuzzy initial value problem with continuous $\widetilde{F}$. For all $\alpha \in [0,1]$ let

$$\mathcal{C}_\alpha := \{\mathsf{f} : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n \,|\, \mathsf{f} \in \mathcal{C} \wedge \mathsf{f} \text{ is a selection of } F_{1-\alpha}\} \subset \mathcal{C} \ .$$

Obviously, we have $\mathcal{C}_\beta \subset \mathcal{C}_\alpha$ for $\beta \leq \alpha$. As already mentioned above, $\mathcal{C}_\alpha$ is thought of as a confidence region for the unknown function f_0. Our objective is to guarantee the existence of a probability measure on the set $\mathcal{C}$ of Carathéodory functions compatible with the restrictions imposed by these confidence regions. Since we may have $\mathcal{C}_\alpha = \mathcal{C}_\beta$ for $\beta \neq \alpha$, we cannot require $\text{Prob}(\mathsf{f}_0 \in \mathcal{C}_\alpha) = \alpha$ for all $0 \leq \alpha \leq 1$. Rather, the probability associated with a set $\mathcal{C}_\alpha$ should be defined as $\tilde{\mu} = \tilde{\mu}(\mathcal{C}_\alpha) = \max\{\delta \in [0,1] \,|\, \mathcal{C}_\alpha = \mathcal{C}_\delta\}$. Therefore, let $\mathcal{A}' := \{\mathcal{C}_\alpha \,|\, \alpha \in [0,1]\}$ and $A := \{\tilde{\mu}(\mathcal{C}') \,|\, \mathcal{C}' \in \mathcal{A}'\}$. Observe that the assumptions concerning the values of $\widetilde{F}$ guarantee the existence of the maximum in the definition of $\tilde{\mu}(\mathcal{C}_\alpha)$.

$\mathcal{A}'$ is the set of confidence regions associated with the fuzzy function $\widetilde{F}$. For this interpretation to make sense, we have to show the existence of an underlying probability space $(\mathcal{C}_1, \mathcal{A}, \mu)$ such that $\mathcal{A}' \subset \mathcal{A}$ and $\mu(\mathcal{C}') = \tilde{\mu}(\mathcal{C}')$ for all $\mathcal{C}' \in \mathcal{A}'$.

Proposition 2 *Let $\mathcal{A} = \sigma(\mathcal{A}') \subset 2^{\mathcal{C}_1}$ be the σ-algebra generated by $\mathcal{A}'$. There is a probability measure μ on $(\mathcal{C}_1, \mathcal{A})$ such that $\mu(\mathcal{C}') = \tilde{\mu}(\mathcal{C}')$ for all $\mathcal{C}' \in \mathcal{A}'$.*

Proof. The ring $\mathcal{R}$ generated by $\mathcal{A}'$ is the set of all $\mathcal{D} \subset \mathcal{C}_1$ which can be written as

$$\mathcal{D} = \mathcal{D}(\alpha_1, \beta_1) \cup \ldots \cup \mathcal{D}(\alpha_m, \beta_m) \cup \mathcal{D}_{m+1} \ ,$$

where $m \in \mathbb{N}_0$, $\alpha^* \leq \alpha_1 < \beta_1 < \alpha_2 < \ldots < \alpha_m < \beta_m \leq 1$, $\alpha_k, \beta_k \in A$, $\mathcal{D}(\alpha, \beta) := \mathcal{C}_\beta \setminus \mathcal{C}_\alpha$, and $\mathcal{D}_{m+1} = \emptyset$ or $\mathcal{D}_{m+1} = \mathcal{C}_{\alpha^*}$, where $\alpha^* := \min\{\alpha \,|\, \alpha \in A\}$. The extension of $\tilde{\mu} : \mathcal{A}' \rightarrow [0,1]$ to $\mathcal{R}$, again denoted $\tilde{\mu}$, is given by

$$\tilde{\mu}(\mathcal{D}) := (\beta_1 - \alpha_1) + \ldots + (\beta_m - \alpha_m) + \alpha^* \left(1 - \chi_{\{\emptyset\}}(\mathcal{D}_{m+1})\right) .$$

Since $\tilde{\mu}$ is a pre-measure on $\mathcal{R}$, it can be extended to a probability measure μ on $\mathcal{A} = \sigma(\mathcal{R}) = \sigma(\mathcal{A}')$. $\Box$.

Remark 1 *So far, we have ignored the uncertain information $\widetilde{X}_0$ concerning the initial system state, i.e., we principally assumed $\widetilde{X}_0$ to be a crisp system state $\mathsf{x}_0 \in \mathbb{R}^n$ in Proposition 2. However, it is not difficult to generalize this result to guarantee the existence of a probability measure on $(\mathcal{C}_1 \times D_1, \sigma(\mathcal{A}''))$. Here, $D_1 := \overline{\mathrm{supp}}(\widetilde{X}_0)$, and $\mathcal{A}''$ is defined as a set of elements $\mathcal{C}_\alpha \times D_\alpha$ in the same way as $\mathcal{A}'$.*

According to the definition of $X_\alpha(t)$ and the fact that $\mathrm{Prob}(\mathsf{f}_0 \in \mathcal{C}') = \tilde{\mu}(\mathcal{C}')$ for all $\mathcal{C}' \in \mathcal{A}'$, we have

$$\mathrm{Prob}(\mathsf{x}_0(t) \in X_\alpha(t)) \geq \tilde{\mu}(\mathcal{C}_{1-\alpha}) \geq 1 - \alpha \,. \tag{13}$$

The $\geq$ relation in (13) can be seen as a general implication of our "set-valued" approach to modelling of uncertain dynamics: Our estimation of the system behavior is an outer estimation in the sense that the whole class $\mathcal{F}$ of functions associated with a set-valued function F by means of (12) is taken into account. However, the class $\mathcal{F}'$ of functions we actually have in mind as possible candidates for f_0 might be smaller than $\mathcal{F}$. This *interpretation problem* can be formulated as follows: Let $\gamma(\mathcal{F}')$ denote the set-valued function F associated with a class $\mathcal{F}'$ of functions, i.e., $F(t, \mathsf{x}) = \{\mathsf{f}(t, \mathsf{x}) \,|\, \mathsf{f} \in \mathcal{F}'\}$. Moreover, denote by $\gamma'(F)$ the class of functions associated with a set-valued function by means of (12). Then, we generally have $\mathcal{F}' \subset \gamma'(\gamma(\mathcal{F}'))$, but $\mathcal{F}' \neq \gamma'(\gamma(\mathcal{F}'))$. In the context of probabilistic interpretations of fuzzy functions this problem appears in the following way: In passing from a probability measure μ_0 on $\mathcal{C}_1$ to a model in form of a fuzzy function, i.e., a system $\mathcal{A}'$ of confidence regions, some information is lost. Proposition 2 only guarantees the existence of a probability measure μ compatible with the constraints imposed by $\mathcal{A}'$. However, since this measure need not be uniquely defined, it is generally not possible to recover μ_0 from $\mathcal{A}'$.

Example 1 *Let $\mathcal{F}' := \{f : \mathbb{R} \rightarrow \mathbb{R} \,|\, f \equiv c, c \in [-1,1]\}$ be a parameterized family of functions. Then, $F := \gamma(\mathcal{F}') \equiv [-1,1]$ and $\sin(t) \in \mathcal{F} = \gamma'(F)$, although $\sin(t) \notin \mathcal{F}'$. Thus, $-\cos(t)$ is an element of the set $\mathcal{X}$ of possible solutions to the initial value problem $\dot{x}(t) \in F(x(t))$, $x(0) = -1$. However, the set of solutions to the problem $\dot{x} \in \mathcal{F}'$ (the function $\dot{x} : [0, \infty) \rightarrow \mathbb{R}$ is an element of $\mathcal{F}'$) and $x(0) = -1$ is given by*

$$\mathcal{X}' = \{x : [0, \infty) \rightarrow \mathbb{R} \,|\, x(t) = c\,t - 1, c \in [-1,1]\} \,.$$

Obviously, $\mathcal{X}' \subsetneq \mathcal{X}$. Observe that $\mathcal{X}$ is compatible with the assumption of a uniform probability measure μ on $\mathcal{F}$ in the probabilistic setting, whereas $\mathcal{X}'$ is associated with a measure μ' concentrated on $\mathcal{F}' \subset \mathcal{F}$.

4.3. *Properties of Reachable Sets*

As already mentioned above, we would like to characterize the behavior of a generalized or "fuzzy" dynamical system by means of reachable sets. We are now going to consider some properties of these sets. We suppose $F : [0,T] \times \mathbb{R}^n \to 2^{\mathbb{R}^n}$ to satisfy the following:

(a) F has nonempty, compact and convex values.

(b) F is continuous in t (with respect to d_H.)

(c) The following Lipschitz condition is satisfied:

$$d_H(F(t,\mathsf{x}), F(t,\mathsf{y})) \leq L\,|\mathsf{x}-\mathsf{y}|$$

for all $\mathsf{x},\mathsf{y} \in \mathbb{R}^n$ with a (global) Lipschitz constant $L > 0$.

(d) F is bounded.

The following proposition follows directly from Corollary 7.1 and Corollary 7.2 in [12].

Proposition 3 *Under assumptions (a), (b), and (c) the reachable sets $X(t) \subset \mathbb{R}^n$ are compact and connected.*

In order to guarantee convexity of $X(t)$, additional assumptions have to be made which are seldom satisfied, such as concavity of F [7], i.e.,

$$\alpha F(t,\mathsf{x}_1) + \beta F(t,\mathsf{x}_2) \subset F(t, \alpha\,\mathsf{x}_1 + \beta\,\mathsf{x}_2) \tag{14}$$

for all $\alpha, \beta > 0$, $\alpha + \beta = 1$.

Since one is often interested in classical solutions, it is interesting to compare the set of solutions (absolutely continuous functions satifying (7) almost everywhere) with the set of classical solutions of (7), i.e., those functions $\mathsf{x} : [t_0, T] \to \mathbb{R}^n \in C^1[t_0,T]$ satisfying (7). According to the following theorem, which is essentially Theorem 7 in [17], the reachable sets associated with $\mathcal{X}$ resp. $\mathcal{X}'$ are "almost identical."

Theorem 2 *Let F satisfy (a), (b), and (c). Then, the set $X'(t) \subset \mathbb{R}^n$ reachable by classical solutions at time $t \in [t_0, T] \subset \mathbb{R}$ is a dense subset of the reachable set $X(t)$.*

4.4. *Properties of Fuzzy Reachable Sets*

Proposition 4 *Suppose the fuzzy function $\widetilde{F} : J \times \mathbb{R}^n \to \mathcal{E}^n$ to be continuous in t and to satisfy a Lipschitz condition*

$$\tilde{d}_H(\widetilde{F}(t,\mathsf{x}), \widetilde{F}(t,\mathsf{y})) \leq L\,|\mathsf{x}-\mathsf{y}|$$

on $J \times \mathbb{R}^n$ with a Lipschitz constant $L > 0$. Consider the set $\widetilde{\mathcal{X}}$ of solutions to (10). The reachable set $\widetilde{X}(t)$ associated with $\widetilde{\mathcal{X}}$ is a normal, upper semicontinuous, and compactly supported fuzzy set for all $t \in [0,T]$. If $\widetilde{F}$ is also concave, then $\widetilde{X}(t) \in \mathcal{E}^n$.

Proof. Since $\widetilde{F}$ is normal and continuous on $J \times \mathbb{R}^n$, a continuous selection $\mathsf{f} : J \times \mathbb{R}^n \to \mathbb{R}^n$ exists such that $\mathsf{f}(t, \mathsf{u}) \in [\mu_{\widetilde{F}(t,\mathsf{u})}]_1$ on $J \times \mathbb{R}^n$. Furthermore, $[\mu_{\widetilde{X}_0}]_1 \neq \emptyset$. Thus, a solution $\mathsf{x}(t)$ to (11) with $\alpha = 1$ exists (see basic existence theorems), and this solution belongs to $\mathcal{X}_1$, which means that $X_1(t) \neq \emptyset$ and $\widetilde{X}(t)$ is normal. The assumptions concerning $\widetilde{F}$ guarantee the reachable set $\widetilde{X}(t)$ to be contained in a bounded set $B \subset \mathbb{R}^n$ for all $t \in [0, T]$. Since the elements $\widetilde{A} \in \mathcal{E}^n$ are compact and convex, compactness of $X_\alpha(t)$ for all $\alpha \in [0, 1]$ follows from Corollary 7.2 in [12]. The fact that $X_\alpha(t)$ is convex under the additional assumption (14) is Theorem 14 in [7]. $\square$.

Proposition 4 shows that $\widetilde{X}(t)$ has some nice properties under reasonable assumptions. Nevertheless, this fuzzy set will generally have a rather complicated structure. For example, since concavity of F_α is seldom satisfied, the α-cuts of $\widetilde{X}(t)$ will usually not be convex. It should also be clear that we cannot expect $\widetilde{X}(t)$ to belong to special classes of fuzzy sets, such as Gaussian, although this may be the case for the values $\widetilde{F}(t, \mathsf{x})$ and $\widetilde{X}_0$. Furthermore, already the approximation of crisp sets $Y \subset \mathbb{R}^n$ turned out to be complicated, especially for higher dimensions. Due to this reason, we will only treat the (set-valued) problem of finding the reachable sets $X_\alpha(t)$ of problem (11) for different values of α. Given reachable sets $X_{\alpha_1}(t), \ldots, X_{\alpha_m}(t)$, the fuzzy set $\widetilde{X}(t)$ is then approximated by means of

$$\mu_{\widetilde{X}(t)}(\mathsf{x}) = \max\{\alpha_k \,|\, \mathsf{x} \in X_{\alpha_k}(t), 1 \le k \le m\} \ .$$

In this context, it is interesting to know in which way $X_\alpha(t)$ depends on α.

Lemma 2 *Let $\widetilde{A} \in \mathbb{F}(\mathbb{R}^n)$ be some fuzzy set and suppose $\alpha_n \to \alpha$ as $n \to \infty$ for some sequence $(\alpha_n) \subset [0, 1]$. Define $A_n := [\mu_{\widetilde{A}}]_{\alpha_n}$ and $A := [\mu_{\widetilde{A}}]_\alpha$. Then,*

$$\beta(A_n, A) \to 0 \quad as \quad n \to \infty \ . \tag{15}$$

Proof. Suppose (15) not to hold. Then, some $\delta > 0$ exists such that for each $n \in \mathbb{N}$ we can find $j(n) \ge n$ and $\mathsf{a}_n \in A_{j(n)}$ satisfying $\rho(\mathsf{a}_n, A) > \delta$. Since $\widetilde{A}$ is compactly supported, (a_n) is bounded, i.e., it has a convergent subsequence $(\mathsf{a}_k) \subset B := \mathbb{R}^n \setminus (A + B_\delta(0))$, where $B_\delta(0) := \{\mathsf{x} \in \mathbb{R}^n \,|\, |\mathsf{x}| < \delta\}$. Observe that B is closed, and recall that $\mu_{\widetilde{A}}(\mathsf{a}_n) \ge \alpha_{j(n)}$ for $n \in \mathbb{N}$. Thus, we have $\mathsf{b} := \lim_{k\to\infty} \mathsf{a}_k \in B$ and $\mu_{\widetilde{A}}(\mathsf{b}) \ge \alpha$, which is a contradiction. $\square$.

Lemma 3 *Consider a fuzzy set $\widetilde{A}$ with membership function $\mu_{\widetilde{A}} : \mathbb{R}^n \to [0, 1]$ such that*

$$\mu_{\widetilde{A}}(\mathsf{x}) < 1 \Rightarrow \forall\, \delta > 0\, \exists\, \mathsf{y} \in \mathbb{R}^n \,:\, |\mathsf{x} - \mathsf{y}| < \delta \wedge \mu_{\widetilde{A}}(\mathsf{y}) > \mu_{\widetilde{A}}(\mathsf{x}) \tag{16}$$

for all $\mathsf{x} \in \operatorname{supp}(\widetilde{A})$. Now, consider some sequence $(\alpha_n) \subset [0, 1]$, $\alpha_n \to \alpha \in [0, 1]$. Let $C_n := [\mu_A]_{\alpha_n}$ and $C := [\mu_A]_\alpha$. Then $d_H(C_n, C) \to 0$ as $n \to \infty$.

Proof. $\beta(C_n, C) \to 0$ follows from Lemma 2. For $\mathsf{x} \in C$ and $\delta > 0$ some value $\mathsf{y} \in \mathbb{R}^n$ exists such that $|\mathsf{x} - \mathsf{y}| < \delta$ and $\mu_{\widetilde{A}}(\mathsf{y}) > \mu_{\widetilde{A}}(\mathsf{x}) \ge \alpha$. Thus, since $\alpha_n \to \alpha$, we have $\mathsf{y} \in C_n$ for almost all $n \in \mathbb{N}$. Now, suppose $\beta(C, C_n) \not\to 0$. Then, some

$\delta > 0$ exists such that for all $n \in \mathbb{N}$ we can find $j(n) \geq n$ and $\mathsf{x}_n \in C$ satisfying $\rho(\mathsf{x}_n, C_{j(n)}) > \delta$. Since C is compact, (x_n) has a convergent subsequence (x_k) with $\mathsf{x}_k \to \mathsf{x} \in C$, which contradicts the fact that $\rho(\mathsf{x}, C_n) \to 0$. Therefore, $\beta(C, C_n) \to 0$ and hence $d_H(C_n, C) \to 0$ as $n \to \infty$. $\Box$.

Remark 2 *Property (16) is satisfied if $\mu_{\widetilde{A}}$ is continuous on $\mathbb{R}^n$ and differentiable on $B := \mathrm{supp}(\widetilde{A}) \setminus \{\mathsf{x} \in \mathbb{R}^n \,|\, \mu_{\widetilde{A}}(\mathsf{x}) = 1\}$, and $\mathrm{grad}\,\mu_{\widetilde{A}}(\mathsf{x}) \neq 0$ on B.*

Proof. Since $\mu_{\widetilde{A}}$ is continuous, B is open. Now, suppose (16) does not hold for $\mathsf{x} \in B$, i.e. some $\delta > 0$ exists such that $B_\delta(\mathsf{x}) \subset B$ and $\mu_{\widetilde{A}}(\mathsf{x}) \geq \mu_{\widetilde{A}}(\mathsf{y})$ for all $\mathsf{y} \in B_\delta(\mathsf{x})$. But in this case x is a local maximum of $\mu_{\widetilde{A}}$ in contrast to the assumption that $\mathrm{grad}\,\mu_{\widetilde{A}}(\mathsf{x}) \neq 0$. $\Box$.

Proposition 5 *Consider problem (10) and let F_α be upper semicontinuous in x and $|F_\alpha(t, \mathsf{x})| \leq m(t)$ on $J \times \mathbb{R}^n$ with $m(t)$ integrable. Then, the function*

$$G : [0,1] \to 2^{\mathbb{R}^n} \setminus \{\emptyset\}\,,\ \alpha \mapsto X_\alpha(t)$$

is upper semicontinuous for all $t \in J$.

Proof. From Lemma 2 follows that

$$\beta([\mu_{\widetilde{X}_0}]_{\alpha_i}, [\mu_{\widetilde{X}_0}]_\alpha) \to 0 \text{ and } \beta(F_{\alpha_i}(t, \mathsf{x}), F_\alpha(t, \mathsf{x})) \to 0$$

for all $(t, \mathsf{x}) \in J \times \mathbb{R}^n$ as $\alpha_i \to \alpha$. Since $[\mu_{\widetilde{X}_0}]_\lambda$ and $[\mu_{\widetilde{F}(t,\mathsf{x})}]_\lambda$ are also nonempty, closed, and convex for all $\lambda \in [0,1]$, the proposition follows from the corollary to Lemma 2 in [7]. $\Box$.

Proposition 6 *Consider problem (10) and suppose the fuzzy sets $\widetilde{X}_0$ and $\widetilde{F}(t, \mathsf{x})$ to satisfy (16). Furthermore, let F_α be continuous in t and satisfy a Lipschitz condition*

$$\tilde{d}_H(\widetilde{F}(t, \mathsf{x}), \widetilde{F}(t, \mathsf{y})) \leq k(t)|\mathsf{x} - \mathsf{y}|, \quad k(t) \in L^1 \ .$$

Then, the function

$$G : [0,1] \to 2^{\mathbb{R}^n} \setminus \{\emptyset\}\,,\ \alpha \mapsto X_\alpha(t)$$

is continuous for each $t \in J$.

Proof. From Lemma 3 follows that $[\mu_{\widetilde{X}_0}]_{\alpha_i} \to [\mu_{\widetilde{X}_0}]_\alpha$ and $F_{\alpha_i} \to F_\alpha$ as $\alpha_i \to \alpha$. Since $[\mu_{\widetilde{X}_0}]_\lambda$ and $[\mu_{\widetilde{F}(t,\mathsf{x})}]_\lambda$ are also nonempty and closed for each $\lambda \in [0,1]$, the proposition follows from Theorem 9 in [7]. $\Box$.

5. Implementation and Example

Since it is generally not possible to find analytically the reachable sets associated with a generalized or fuzzy initial value problem, we have investigated methods for finding numerical approximations of these sets. In this section, we restrict ourselves to a brief description of the basic ideas and a simple example illustrating this approach.

Principally, the numerical methods are based on two kinds of discretization. The time interval $[0, T]$ is replaced by a grid $0 = t_0 < t_1 < \ldots < t_N = T$ with stepsize

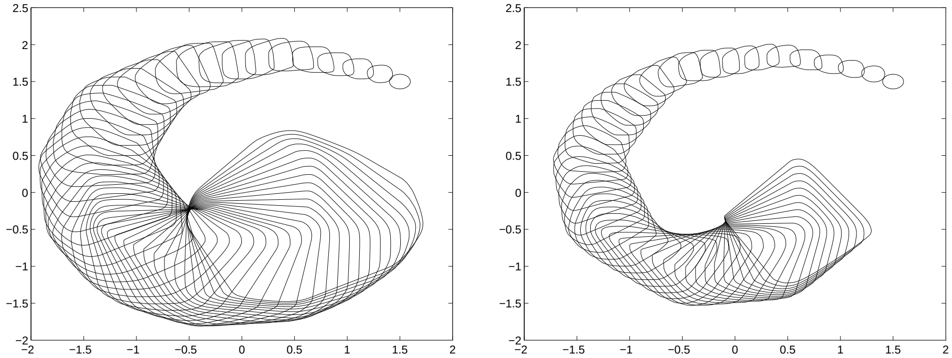

Figure 1: The α-cuts $X_\alpha(t)$ ($\alpha = 0.7$ (left), $\alpha = 0.9$ (right), $t \in \{0, 0.1, \ldots, 5\}$) of the reachable fuzzy sets $\widetilde{X}(t)$.

$\Delta t = T/N = t_i - t_{i-1}$ ($i = 1, \ldots, N$). A simple first order discretization of (7) is then given by

$$\frac{\mathsf{y}_{i+1} - \mathsf{y}_i}{\Delta t} \in F(t_i, \mathsf{y}_i) \ . \tag{17}$$

Based on (17) the following generalized Euler scheme can be defined:

$$Y(t_{i+1}) := \bigcup_{\mathsf{y} \in Y(t_i)} \mathsf{y} + \Delta t \, F(t_i, Y(t_i)) \, , \quad Y(0) := X_0 \ . \tag{18}$$

Since the sets $Y(t_i)$ may have very complicated structures, it is generally not possible to represent them exactly. Thus, in addition to a discretization with regard to time, we have to perform a second approximation in the form of a discretization of $2^{\mathbb{R}^n}$. For this purpose, consider a class $\mathcal{A} \subset 2^{\mathbb{R}^n}$ of sets which can be represented by means of a certain finite data structure. Denote by $\mathcal{A}(Y) \in \mathcal{A}$ the approximation of a set $Y \subset \mathbb{R}^n$. We can then define the following approximation of (18):

$$Z(t_{i+1}) := \mathcal{A}\left(\bigcup_{\mathsf{z} \in Z(t_i)} \mathsf{z} + \Delta t \, \mathcal{A}(F(t_i, \mathsf{z})) \right) \, , \quad Z(0) := \mathcal{A}(X_0) \ . \tag{19}$$

The implementation of our method is based on the iteration scheme (19). The class $\mathcal{A}$ used in (19) for approximating sets was implemented as different classes of geometrical bodies, such as convex hulls or more general classes including non-convex sets. Under certain conditions, approximations $Z(t)$ of reachable sets $X(t)$ can be found with any degree of accuracy for all $t \in [0, T]$.

Consider the following example as an illustration of our approach. This example combines the methods for modelling uncertain functional relationships with the simulation methods discussed in this paper. The behavior of a two-dimensional (autonomous) system was charcterized by 25 linguistic rules of the form "If x is `small` and y is `small` then $\dot{x}$ is `large` and $\dot{y}$ is `small`." The set of five linguistic variables used in this example is modelled by Gaussian membership functions. We associate a fuzzy function $\widetilde{F}$ with this set of rules based on the method presented in Section 2. This function $\widetilde{F}$ then serves as the right hand side of our fuzzy initial value problem. The information that the initial system state is `approximately` $(3/2, 3/2)$ is modelled as a "fuzzy circle" with center $(3/2, 3/2)$. The α-sections of some fuzzy reachable sets characterizing the behavior of this system are shown in Figure 1 for different values of α. The set of all α-sections of the reachable sets $\widetilde{X}(t)$ forms a "funnel" bounding the true but unknown system trajectory with a certain probability.

6. Related Methods

Contributions to modelling and simulation of uncertain dynamics have already been made from different directions. Next to well-known probabilistic methods, such as stochastic differential equations, approaches based on "alternative" representations of uncertainty have been considered. Particularly, corresponding methods can be found in the research fields of qualitative reasoning and the theory of fuzzy sets. This section should serve as a brief overview of such approaches. A detailed discussion and comparison of different methods is beyond the scope of this paper.

The first qualitative simulation algorithm has been proposed by Kuipers [35]. This algorithm derives a set of qualitative system behaviors compatible with a differential equation, the right hand side of which is restricted by a set of algebraic, derivative, and monotonicity constraints. Similar methods have been proposed in [20,38]. However, these approaches often produce "spurious" system behaviors. A main reason for imprecise predictions is the "merging" of different (quantitative) system behaviors, a general problem of approaches based on a discretization of the state space. Such problems lead to the incorporation of numerical information [6]. Moreover, so-called semi-quantitative approaches [8,29,42] with continuous state space have been considered. These methods make use of interval functions for modelling uncertain functional relationships. However, the corresponding numerical methods generally produce very inaccurate predictions of the system behavior [22]. In connection with qualitative reasoning, we should also mention approaches to modelling and simulation based on other methodologies, such as inductive reasoning [10,11]. Particularly, this last approach is interesting from a modelling point of view: A model is not defined explicitely, but infered implictely based on a set of observations. In a certain sense, this "data-driven" method can be seen as the opposite of model-based approaches.

It is interesting to compare our approach with a special type of fuzzy differential equations proposed in [26,27,37]. These approaches to solving differential equations with "fuzzy" right hand side are principally based on the "fuzzification" of the differential operator. The definition of this operator makes use of a generalization of the so-called Hukuhara difference of sets $X, Y \subset \mathbb{R}^n$: A fuzzy set $\widetilde{Z} \subset \mathbb{F}(\mathbb{R}^n)$ is called the H-difference of $\widetilde{X}$ and $\widetilde{Y}$, denoted $\widetilde{X} - \widetilde{Y}$, if $\widetilde{X} = \widetilde{Y} + \widetilde{Z}$. Here, $+$ is the usual addition of fuzzy sets. A fuzzy set $\widetilde{F}'(t_0)$ is defined to be the derivative of a fuzzy function $\widetilde{F}$ at t_0 if the limits

$$\lim_{\Delta t \nearrow 0} \frac{\widetilde{F}(t_0 + \Delta t) - \widetilde{F}(t_0)}{\Delta t}\ , \ \lim_{\Delta t \nearrow 0} \frac{\widetilde{F}(t_0) - \widetilde{F}(t_0 - \Delta t)}{\Delta t} \tag{20}$$

exist and are equal to $\widetilde{F}'(t_0)$. A solution to a differential equation based on this kind of derivative is thought of as a single trajectory in the "state space" $\mathbb{F}(\mathbb{R}^n)$. That is, it is interpreted as one single object, whereas our interpretation of solutions to fuzzy initial value problems was that of a (fuzzy) set of "ordinary" functions. Seen from this perspective, (20) may lead to results which are not intuitive. Consider the (crisp) problem $\dot{x} = -x$, $x(0) \in [-1, 1]$ with an unknown initial system state as an example. Since $x(t) = a \exp(-t)$ is the general solution of the initial value problem $\dot{x} = -x$, $x(0) = a$, and a is restricted to values within the interval $[-1, 1]$, we should expect to obtain the solution $x(t) \in [-\exp(-t), \exp(-t)]$. However, the fuzzy function (which is actually a set-valued function) solving this initial value problem in the sense of (20) is $\widetilde{F}(t) = [-\exp(t), \exp(t)]$. Particularly, we have $\mathrm{diam}(\widetilde{F}(t)) \to \infty$ instead of $\mathrm{diam}(\widetilde{F}(t)) \to 0$ as $t \to \infty$.

7. Conclusions

In this paper, we have presented a method for modelling and simulation of uncertain dynamical systems. This approach is motivated by applications in knowledge-based systems. However, since uncertainty and incomplete knowledge is an inherent characteristic of modelling dynamical systems, many other applications can be found. Particularly, this method is attractive as a methodology for modelling and simulation in the so-called "soft sciences."

It should be noted that the methods for modelling fuzzy functions and for the simulation of "fuzzy" dynamical systems are principally independent. Of course, the simulation method can use any kind of (reasonable) fuzzy function, regardless of its origination. Particularly, the right hand side can be defined as a combination of precise and fuzzy functions. Based on other interpretations of the fuzzy systems discussed in Section 2 it is also possible to derive other fuzzy functions [23,24]. Particularly, such a system need not be interpreted as a linguistic model. With regard to the probabilistic interpretation of fuzzy functions presented in Section 4, a set of fuzzy rules can simply be seen as a set of input-output pairs $(\mathsf{x}_k, \widetilde{Y}_k)$. Then, the fuzzy set $\widetilde{Y}_k$ can be interpreted as a system of confidence regions for the unknown value $\mathsf{f}_0(\mathsf{x}_k)$. In this case, the inference procedure realized by an additive fuzzy system is nothing but an "interpolation of confidence regions."

According to our opinion, the clear semantical basis and the precision of the predictions are advantages of the simulation method compared with other approaches. The price for this result is an increased complexity, which is principally caused by the problem of handling complex approximations of reachable sets in n-dimensional space. However, the general algorithm based on iteration scheme (19) can be simplified for special classes of systems [22,25]. The identification of such structures and the parallelization of the simulation methods are central aspects of future research.

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