Chapter S:III

III. Informed Search

Best-First Search Basics

"To enhance the performance of AI's programs, knowledge [about the problem domain, which enables us to guide search into promising directions] *is the power."*

[Feigenbaum 1980]

Best-First Search

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Examples for heuristic functions [S:I Examples for Search Problems] :

- 8-Queens problem. Maximize h_1, the number of unattacked cells.
- 8-Puzzle problem. Minimize h_1, the number of non-matching tiles.

Knowledge on how to achieve this (Maximize. . . , Minimize. . .) is beyond that which is built into the state and operator definitions.

Best-First Search Basics

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Examples for heuristic functions [S:I Examples for Search Problems] :

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Where is heuristic knowledge employed in the formalism of systematic search?

- Greedy Search. Move into the direction of a most promising successor n' of the current node.
- Best-First Search. Move into the direction of a most promising node n, where n is chosen among all nodes encountered so far.

Best-First Search Basics

"The promise of a node is estimated numerically by a heuristic evaluation function $f(n)$ *which, in general, may depend on the description of* n*, the description of the goal, the information gathered by the search up to that point, and most important, on any extra knowledge about the problem domain."*

[Pearl 1984]

Best-First Search Basics

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[Pearl 1984]

The evaluation function f may depend on

1. evaluation of the state information given by n,
2. estimates of the complexity of the remaining problem at n in relation to Γ,
3. evaluations of the explored path to n in the search space graph G,
4. domain specific problem solving knowledge K about G.

$$f = f(n, \Gamma, G, K)$$

Objective is to quantify for a generated, but yet unexpanded node n its potential of guiding the search into a desired direction.

Remarks:

- Node n represents a solution base. Therefore, n gives access to information about a path from s to the state represented by n.
- The remaining problem is the problem of determining a solution path for the state in G given by node n. Using such path as a continuation of the solution base given by n, a solution path for s is given. The complexity estimation of the remaining problem is beyond the information encoded in nodes and edges.
- Knowing that G is Euclidean is an example for domain specific problem solving knowledge. Euclidean distances can be used for estimating remaining path length.
- Evaluation functions are domain dependent. Therefore, these functions (or parts of them) will be provided to search algorithms as parameters.
- We could think of doing even more: An evaluation of a solution base by f could also depend on the explored part of the search space graph G, e.g., the emphasis on specific knowledge in the computation of f could be changed thereby. However, such a dependence would require an update of computed f-values (highly inefficient) each time the explored part of G changes.

Best-First Search Basics

Generic Schema for Best-First Algorithms

... from a solution-base-oriented perspective:

1. Initialize a solution base storage.
2. Loop.
   (a) Select a most promising solution base using an evaluation function f.
   (b) Expand the only unexpanded node in the solution base.
   (c) Extend the solution base by one successor node at a time and save it as a new candidate.
   (d) Determine whether a solution path has been found.

Usage:

- Node expansion is used as basic step.
- Best-First Algorithms are informed versions of `Basic_OR`.

Remarks:

- The schema is further extended by termination tests for failure and success.
- The job of the evaluation function is to make two solution bases comparable and hence to provide an order on them.
- Best-first strategies differ in the evaluation functions they use. Placing restrictions on the computation of these functions will establish a taxonomy of best-first algorithms.
- Even when considering constraint satisfaction problems it makes sense to use best-first algorithms. The paradigm "Small is quick!" follows the idea that low cost values will be assigned to solutions with simple structure and that simple structures can be established in a few steps, i.e., in short time.

Best-First Search Algorithms

Notation for Evaluation Functions

- Evaluation functions f are specifically designed for a state-space G.

  This dependency is usually clear from the context. If not, we will use different names $f, f', \ldots$ to distinguish between evaluation functions for different state spaces.

- An evaluation function f for G uses information on a solution base P for some state s in G (a path in G from s to some other state s', the tip-node of the solution base) and knowledge about G.

$f(P)$ From a state-space graph perspective, function f must have a path parameters P that defines a solution base. As f is specific for G no further information is needed.

$f(n)$ From a back-pointer structure oriented perspective, it is enough to provide a node n as argument of f. The back-pointer path defines the solution base to consider.

- In property definitions for f we take a back-pointer perspective although f should be defined as function on paths of G.

  Nodes and back-pointer paths have to be seen as part of any back-pointer structure that is theoretically constructible and meaningful. These structures are NOT restricted to be back-pointer structures produced by some algorithm at some point in time.

  "For all nodes $n \ldots$" therefore has the same meaning as "For all solution bases P for $s \ldots$".

Best-First Search Algorithms

Notation for Evaluation Functions (continued)

Definition 20 (Evaluation Function f)

Let G be state-space graph.

An *evaluation function* f is a function that assigns values $f(n)$ in an ordered set to paths P in G, where paths P are given as back-pointer paths of nodes n.

- We use the extended real numbers $\overline{\mathbf{R}} = \mathbf{R} \cup \{-\infty, +\infty\}$ and the $\leq$-relation as ordered set.
- Evaluation functions f used in algorithm BF is designed for a specific state-space G. f is, therefore, highly domain dependent.
- Algorithms will usually consider only paths starting in s and states on such paths that are available in its back-pointer structure at some point in time. These values will be denoted by $f(n)$ for some node n.

Best-First Search Algorithms

Basic Principles for an Algorithmization of Best-First Search

Prop$_1(G)$ Required Properties of G for Best-First Search

1. G has *Prop*$_0(G)$ properties.
2. Evaluation function f is defined for G and **assigns cost values to paths in** G.
3. f is computable.
4. When f evaluates a solution bases P_{s-n}, the computed value does not depend on the time of computation.
5. When f evaluates a solution bases P_{s-n}, f estimates optimum cost of solution paths that have P_{s-n} as initial part.
6. A most promising solution base has a minimum f-value in a candidate set.

Task

- Determine a solution paths for s in G.

Algorithmization:

- A most promising solution base is searched *among all solution bases* currently maintained by an algorithm.

Remarks:

- Best-first algorithms for state-space graphs are variants of algorithm Basic-OR. So, the solution bases P_{s-n} under consideration are defined by the states and back-pointers stored with the nodes in OPEN or CLOSED.
- If a dead-end recognition $\perp(n)$ is available, no solution base will be considered that contains an inner node labeled "unsolvable" using $\perp(n)$. A dead end recognition $\perp(.)$ can be integrated in f by setting $f(n) = \infty$ if $\perp(n)$ is true.
- Usually, the evaluation function $f(n)$ is based on a heuristic $h(n)$.

  $h(n)$ estimates the optimum cost of a solution path for the rest problem associated with a node n. Ideally, $h(n)$ should consider the probability of the solvability of the problem at node n.

Best-First Search Algorithms

Algorithm: Basic-BF (Compare BFS, BF, Basic-BF*.)

Input: s. Start node representing the initial state (problem) in G.
successors(n). Returns *new instances of* nodes for the successor states in G.
$\star(n)$. Predicate that is *True* if n represents a goal state in G.
constraints(n). Predicate that is *True* if path repr. by n satisfies solution constraints.
$f(n)$. Evaluation function (cost) for the solution base in G represented by n.

Output: A node γ representing a solution path for s in G or the symbol *Fail*.

Basic-BF(s, *successors*, $\star$, *constraints*, f) // A deterministic variant of Basic-OR.

1. s.*parent* = *null*; *add*(s, OPEN, $f(s)$); // Store s on f-sorted OPEN.
2. **LOOP**
3. IF (OPEN == $\emptyset$) THEN RETURN(*Fail*);
4. n = *min*(OPEN, f); // Find most promising (cheapest) solution base.
   remove(n, OPEN); *add*(n, CLOSED);
5. **FOREACH** n' IN *successors*(n) **DO** // Expand n.
   n'.*parent* = n;
   IF $\star(n')$ THEN
   IF *constraints*(n') THEN RETURN(n');
   add(n', OPEN, $f(n')$); // Store n' on f-sorted OPEN.
   ENDDO
6. **ENDLOOP**

Remarks:

- Operationalization of best-first search:
  The function *add*$(n, \texttt{OPEN}, f(n))$ stores a node n according to $f(n)$ in the underlying data structure of the OPEN list. Using a sorted tree (a heap), a node with the minimum f-value is found in logarithmic (constant) time. [OPEN list in DFS] [OPEN list in BFS]

- Since f-values do not change over time, they can be stored with the nodes once computed.

- In all the following algorithms we can make use of dead-end functions $\perp(n)$.

- In addition, memory consumption can be reduced by using *cleanup_closed* in the case of nodes without successors. To save room, we will not include these parts in the pseudocode.

Best-First Search Algorithms

Uniform-Cost Search (UCS) as Variant of `Basic-BF`

Setting:

- The search space graph G contains several solution paths.
- f assigns cost values to solution bases that do not include future cost for extending a solution base to a solution path:

  $f(n)$ = cost of path P_{s-n}

Task:

- Determine a cheapest path from s to some goal $\gamma \in \Gamma$.

Necessary Prerequisite:

- The cost of a solution base is a lower bound for the cheapest solution cost that can be achieved by completing the solution base.

➜ UCS will search G in layers of (nearly) equal cost and
UCS is complete, if G with *Prop*$_0(G)$ is finite + cycle-free, and
UCS will – sometimes – find optimum solution paths in this case.

Remarks:

- Uniform-cost search is also called cheapest-first search.
- A specific cost concept is to assign cost values to edges in search space graphs. A path's cost can be calculated as the sum or as the maximum of the cost values of its edges.

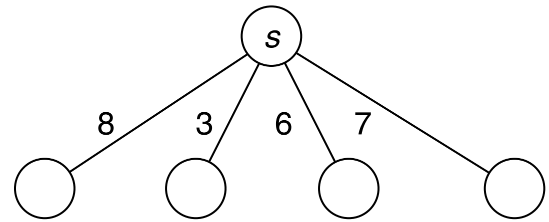
  If edge cost values are limited to non-negative numbers, the path cost of a solution base is an optimistic estimate of a cheapest solution path cost achievable by continuing that solution base.
- Depending on the state-space, the last step to a goal node could be quite expensive. Since delayed termination is not implemented, UCS immediately terminates when finding such a goal node, perhaps returning a suboptimal solution.
- If we have no means to calculate cost values for solution bases or if the cost of a solution base not guaranteed to be a lower bound for the cheapest solution cost that can be achieved by completing the solution base, the algorithm can surely know a minimum cost solution path, only if the set of solution bases in OPEN is exhausted.

Best-First Search Algorithms

Example: Uniform-Cost Search for Optimization

Determine the minimum column sum of a matrix:

8	3	6	7
6	5	9	8
5	3	7	8
1	2	4	6



Best-First Search Algorithms

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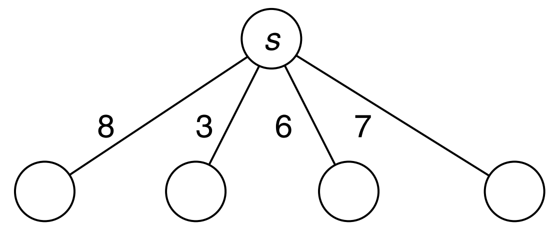
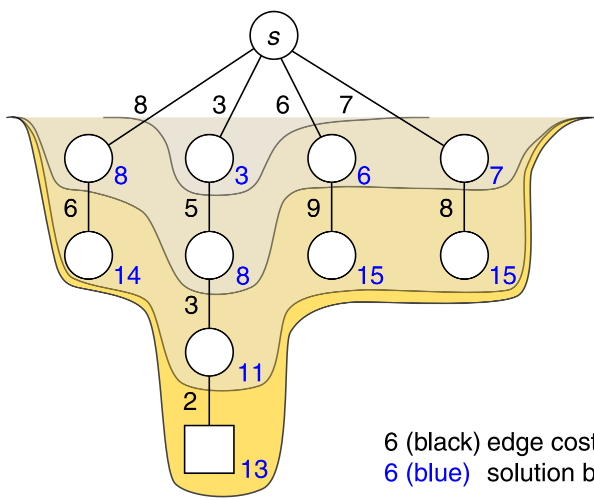
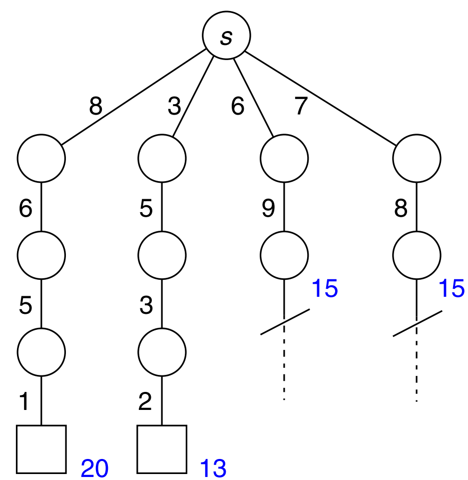


Comparison of UCS (left) and DFS (right):





6 (black) edge cost
6 (blue) solution base cost

Best-First Search Algorithms

Uniform-Cost Search is an uninformed (systematic) search strategy.

Uniform-cost search characteristics:

- Node expansion happens in levels of equal costs:

  A node n with $f(n) = \textit{cost}(n)$ will not be expanded as long as a non-expanded node n' with $f(n') = \textit{cost}(n') < \textit{cost}(n) = f(n)$ resides on the OPEN list.

  ≈ UCS can be seen as application of the BFS strategy to solve optimization problems (using cost instead of depth).

  ≈ BFS can be seen as a UCS variant using $f(n) = \textit{depth}(n)$.
  DFS can be seen as a UCS variant using $f(n) = -\textit{depth}(n)$.

- The optimistic cost estimation is crucial also for the correctness of the Uniform-Cost Search algorithm:

  If the cheapest solution cost that can be achieved by completing the solution base is overestimated we might miss an optimum cost solution path.

Best-First Search Algorithms

Delayed Termination: Basic-BF for Optimization

In general, the first solution found by algorithm Basic-BF may not be optimum with respect to the evaluation function f.

Important preconditions for (provably) finding optimum solution paths:

1. The cost estimate underlying f must be optimistic, i.e., underestimating costs or overestimating merits.

   In particular, the true cost $f_{P_{s-\gamma}}(\gamma)$ of a cheapest solution path $P_{s-\gamma}$ extending a solution base P_{s-n} exceeds its f-value: $f_{P_{s-\gamma}}(\gamma) \geq f_{P_{s-n}}(n)$ (➜ domain-dependent).

2. The termination in case of success ($\star(n) =$ *True*) must be delayed.

   In particular, there is no termination test when reaching a node, but each time when choosing a node from the OPEN list (➜ easily implemented).

➜ Algorithms using delayed termination are indicated by a star (*), Basic-BF becomes Basic-BF*.

Best-First Search Algorithms

Algorithm: Basic-BF* (Compare BF*.)

Input: s. Start node representing the initial state (problem) in G.
successors(n). Returns *new instances of* nodes for the successor states in G.
$\star(n)$. Predicate that is *True* if n represents a goal state in G.
$f(n)$. Evaluation function (cost) for the solution base in G represented by n.

Output: A node γ representing an (optimum) solution path for s in G or the symbol *Fail*.

```
Basic-BF*(s, successors, ⋆, f)   // A delayed termination variant of Basic-BF.

  1.  s.parent = null;  add(s, OPEN, f(s));
  2.  LOOP
  3.    IF (OPEN == ∅) THEN RETURN(Fail);
  4.    n = min(OPEN, f);
  ➜     IF ⋆(n) THEN RETURN(n);   // Delayed termination.
        remove(n, OPEN);  add(n, CLOSED);
  5.    FOREACH n' IN successors(n) DO   // Expand n.
          n'.parent = n;
  ➜       IF ⋆(n') THEN RETURN(n');   // Early termination removed.
          add(n', OPEN, f(n'));
        ENDDO
  6.  ENDLOOP
```

Remarks:

- If the evaluation function f depends on the evaluations of the explored part G of the search space graph ONLY, f is uninformed and algorithm Basic-BF* performs a uniform-cost search with delayed termination.
- In the problem "minimum column sum of a matrix" the evaluation function $f(n)$ which returns the sum of column entries up to n is optimistic if matrix entries are nonnegative.

  In this case, algorithm Basic-BF* returns an optimum column.

Best-First Search Algorithms

Space Efficiency of Basic-BF and Basic-BF*

Approach:
Instead of storing all known paths to a node, only a most promising one is kept.

An implementation of this principle is called path discarding (aka parent discarding).

→ Basic-BF with path discarding is called BF,
BF with delayed termination is called BF*.

Important preconditions for (provably) finding optimum solution paths in OR-graphs by best-first algorithms:

1. The cost estimate underlying f must be order-preserving, i.e., a solution base for a node n that is more promising than some other solution base for n will lead to a solution path which is not inferior to solution paths reached by extending the inferior solution base.

2. In particular, cyclic paths should not be considered.

3. When defining a tie breaking strategy for OPEN, goal nodes must be preferred.

Best-First Search Algorithms

Implementing Path Discarding in Basic-BF

BF(s, *successors*, $\star$, f)   // An path discarding variant of Basic-BF.

1. s.*parent* $=$ *null*; *add*(s, OPEN, $f(s)$);
2. **LOOP**
3. IF (OPEN $==\emptyset$) THEN RETURN(*Fail*);
4. $n=$ *min*(OPEN, f); // Find most promising (cheapest) solution base.
   remove(n, OPEN); *add*(n, CLOSED);
5. **FOREACH** n' IN *successors*(n) **DO** // Expand n.
   - n'.*parent* $= n$;
   - IF $\star(n')$ THEN RETURN(n');
   - n'_{old} $=$ *retrieve*(n', OPEN $\cup$ CLOSED); // State of n' already visited?
   - IF (n'_{old} $==$ *null*)
   - THEN // n' not in OPEN or CLOSED: n' refers to a new state.
     - *add*(n', OPEN, $f(n')$);
   - ELSE // n' refers to an already visited state.
     - IF ($f(n') < f(n'_{old})$) // Compare cost of solution bases.
     - THEN // Solution base of n' is cheaper: path discarding.
       - n'_{old}.*parent* $= n'$.*parent*; $f(n'_{old}) = f(n')$;
       - IF $n'_{old} \in$ CLOSED THEN *remove*(n'_{old}, CLOSED); *add*(n'_{old}, OPEN, $f(n'_{old})$); ENDIF
     - ENDIF
   - ENDIF

   ENDDO
6. **ENDLOOP**

Remarks:

- The function *retrieve*$(n', \texttt{OPEN} \cup \texttt{CLOSED})$ retrieves (without removing) a previously stored node instance from OPEN resp. CLOSED referring to the same state in G as n'.

- Due to space limitations the above algorithm does not mention that the new instance of a node n' that has a counterpart in OPEN or CLOSED has to be removed. BF always keeps of all node instances referring to the same state only that one that was generated first.

- Statement $f(n'_{old}) = f(n')$ in algorithm BF is to be understood in the sense that old f-values that have been stored (with the nodes) are overwritten. Not only the new parent reference, also the new f-value is kept.

- The updating of back-pointers performed by BF algorithms preserves the structure of the traversal tree (maintained by BF via nodes stored in OPEN and CLOSED and back-pointers) at any point in time t.
  At each point in time (i.e., each time that the algorithm is at the beginning of the main loop) BF has a traversal tree at hand which is a subtree of G rooted in s.

Remarks:

- Path discarding entails the risk of not finding desired solutions. The risk can be eliminated by restricting to evaluation functions f that fulfill particular properties. Keyword: *Order preserving property* [S:III Specialized Cost Measures]
- If cyclic paths have smaller f-values than corresponding cyclefree paths, the back-pointer structure will be corrupted when a cycle is found.
- As a consequence of path discarding *at most one solution base* for each state in G,
- As a consequence of path discarding, for two paths leading to the same node, the one with the higher f-value is discarded.

Best-First Search Algorithms

Path Discarding for a Node n'

```
5.    FOREACH n' IN successors(n) DO      // Expand n.
        ...
        n'_old = retrieve(n', OPEN ∪ CLOSED);     // State of n' already visited?
        IF ( n'_old == null )
        ...
        ELSE
          IF ( f(n') < f(n'_old) )    // Compare cost of solution bases.
          THEN    // Solution base of n' is cheaper:  path discarding.
            n'_old.parent = n'.parent;  f(n'_old) = f(n');
            IF n'_old ∈ CLOSED THEN remove(n'_old, CLOSED);  add(n'_old, OPEN, f(n'_old)); ENDIF
        ENDIF
```

- $f(n')$ is computed using the new node instance n' and the back-pointer path from s to n' via its parent n.
- $f(n'_{old})$ is computed using the old node instance n' and the back-pointer path from s to n'_{old}.
- n' and n'_{old} are referring to the same state in G.
- Path discarding is performed implicitly by maintaining at most one node instance referring to some state and, therefore, maintaining at most one back-pointer, i.e., at most one path.
- Algorithm BF cannot recover paths that were discarded, i.e., path discarding is irrevocable.
- f-values do not change over time. Once computed, f-values are stored with the nodes.

Best-First Search Algorithms

Re-evaluation of a Node n'

Case 1: n'_{old} is still on OPEN.

```
5.    FOREACH n' IN successors(n) DO      // Expand n.
        ...
        n'_old = retrieve(n', OPEN ∪ CLOSED);    // State of n' already visited?
        IF ( n'_old == null )
        ...
        ELSE
          IF ( f(n') < f(n'_old) )    // Compare cost of solution bases.
          THEN   // Solution base of n' is cheaper: path discarding.
            n'_old.parent = n'.parent;  f(n'_old) = f(n');
            IF n'_old ∈ CLOSED THEN remove(n'_old, CLOSED);  add(n'_old, OPEN, f(n'_old)); ENDIF
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Best-First Search Algorithms

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            n'_old.parent = n'.parent;   f(n'_old) = f(n');
            IF n'_old ∈ CLOSED THEN remove(n'_old, CLOSED);  add(n'_old, OPEN, f(n'_old)); ENDIF
        ENDIF
```
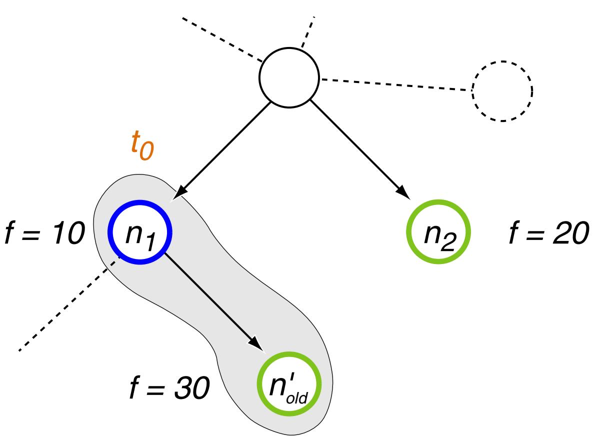
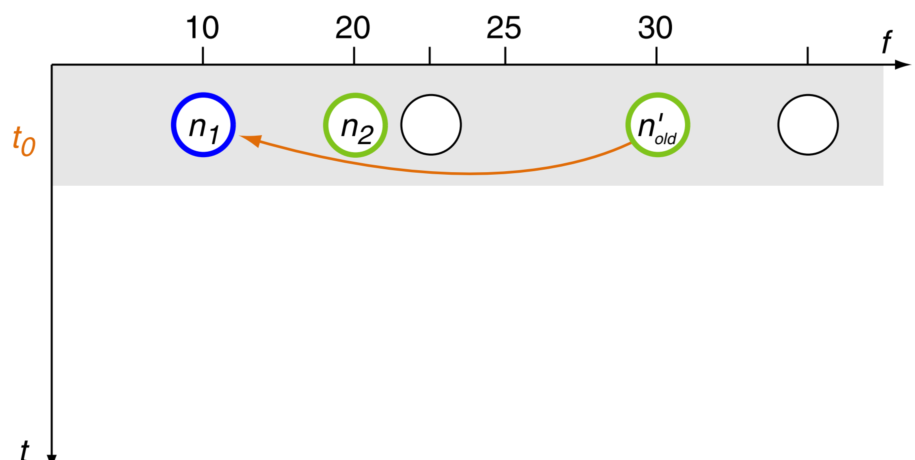

State-space:



OPEN ∪ CLOSED list:



Re-evaluation of a Node n'

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          THEN    // Solution base of n' is cheaper:  path discarding.
            n'_old.parent = n'.parent;  f(n'_old) = f(n');
            IF n'_old ∈ CLOSED THEN remove(n'_old, CLOSED);  add(n'_old, OPEN, f(n'_old)); ENDIF
        ENDIF
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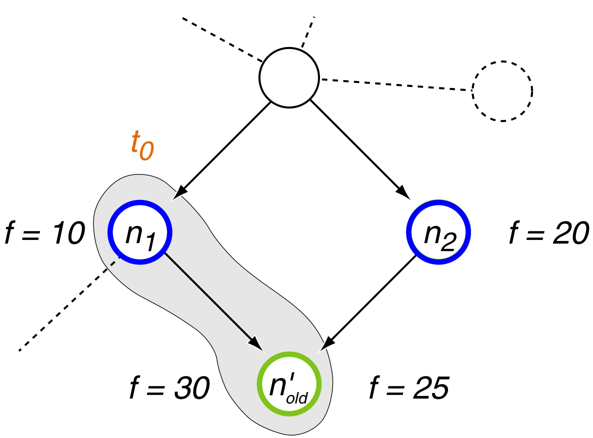
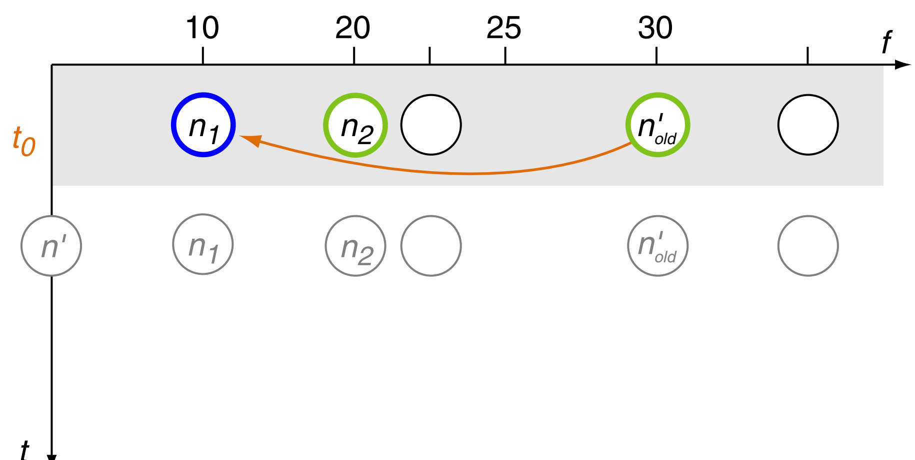
State-space:



OPEN ∪ CLOSED list:



Best-First Search Algorithms

Re-evaluation of a Node n'

Case 1: n'_{old} is still on OPEN.

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        IF ( n'_old == null )
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        ELSE
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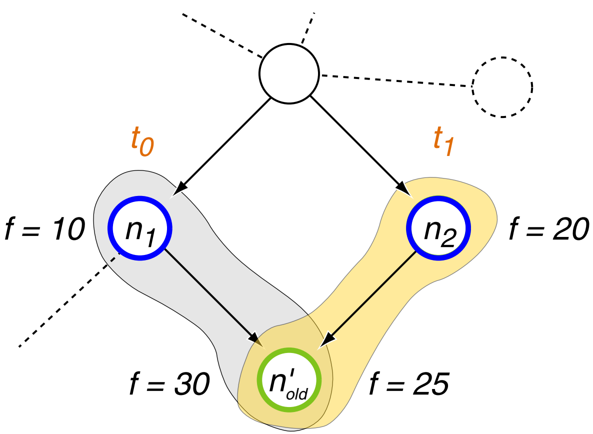
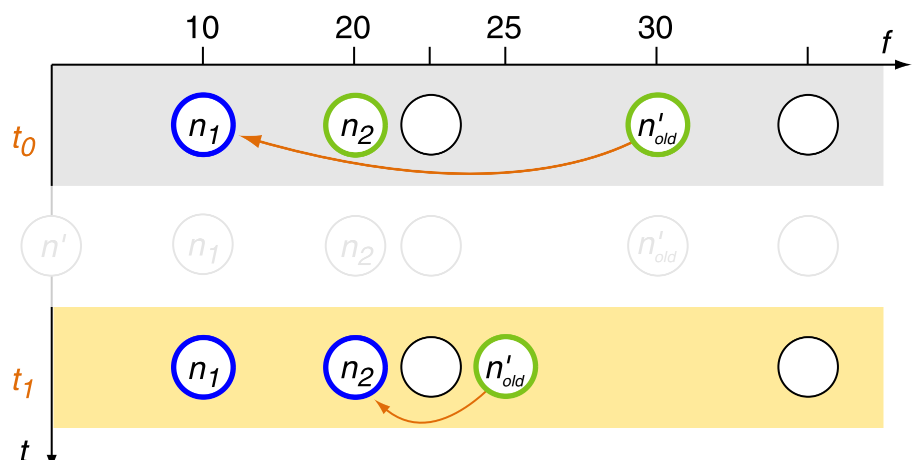
          THEN    // Solution base of n' is cheaper:  path discarding.
            n'_old.parent = n'.parent;  f(n'_old) = f(n');
            IF n'_old ∈ CLOSED THEN remove(n'_old, CLOSED);  add(n'_old, OPEN, f(n'_old)); ENDIF
        ENDIF
```

State-space:



OPEN ∪ CLOSED list:



Best-First Search Algorithms

Re-evaluation of a Node n' (continued)

Case 2: n'_{old} is already on CLOSED.

```
5.    FOREACH n' IN successors(n) DO     // Expand n.
        ...
        n'_old = retrieve(n', OPEN ∪ CLOSED);    // State of n' already visited?
        IF ( n'_old == null )
        ...
        ELSE
          IF ( f(n') < f(n'_old) )    // Compare cost of solution bases.
          THEN   // Solution base of n' is cheaper:  path discarding.
            n'_old.parent = n'.parent;  f(n'_old) = f(n');
            IF n'_old ∈ CLOSED THEN remove(n'_old, CLOSED);  add(n'_old, OPEN, f(n'_old)); ENDIF
        ENDIF
```

Best-First Search Algorithms

Re-evaluation of a Node n' (continued)

Case 2: n'_{old} is already on CLOSED.

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        ...
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          IF ( f(n') < f(n'_old) )    // Compare cost of solution bases.
          THEN    // Solution base of n' is cheaper:  path discarding.
            n'_old.parent = n'.parent;  f(n'_old) = f(n');
            IF n'_old ∈ CLOSED THEN remove(n'_old, CLOSED);  add(n'_old, OPEN, f(n'_old)); ENDIF
        ENDIF
```
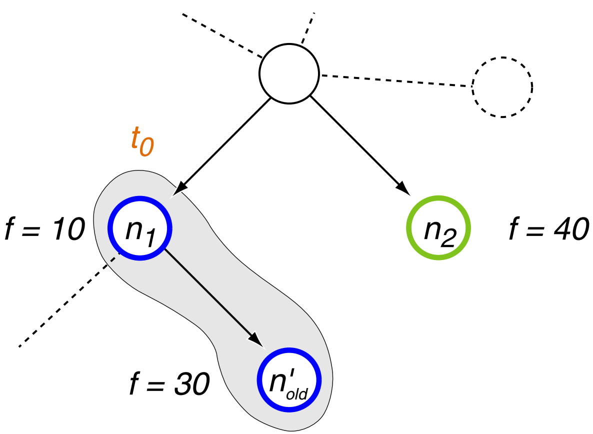
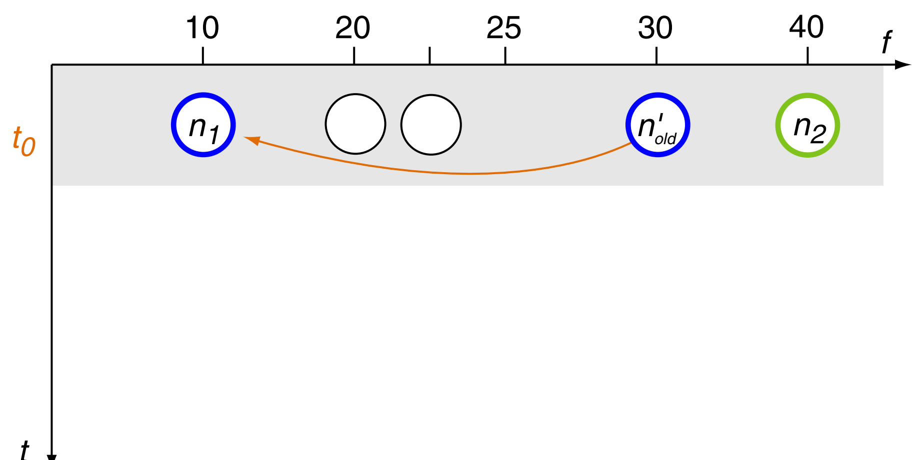

State-space:



OPEN ∪ CLOSED list:



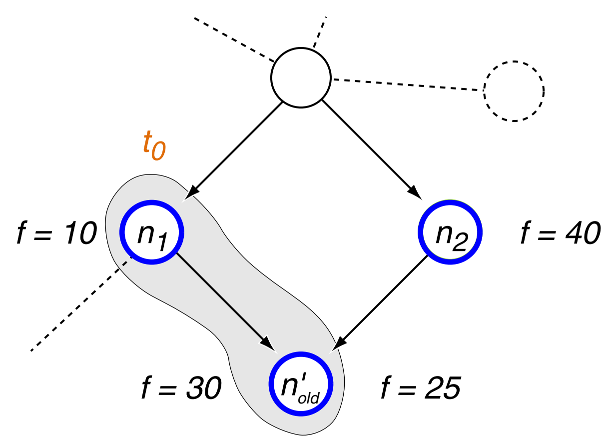
Best-First Search Algorithms

Re-evaluation of a Node n' (continued)

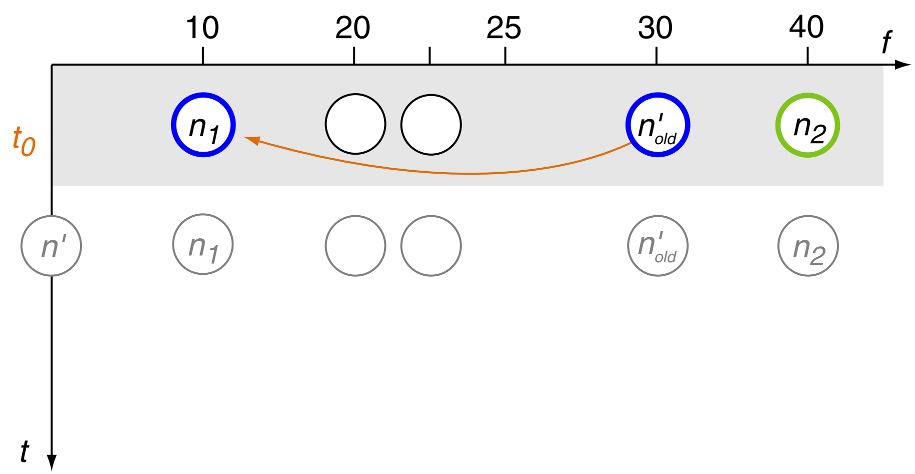
Case 2: n'_{old} is already on CLOSED.

```
5.     FOREACH n' IN successors(n) DO     // Expand n.
         ...
         n'_old = retrieve(n', OPEN ∪ CLOSED);    // State of n' already visited?
         IF ( n'_old == null )
         ...
         ELSE
           IF ( f(n') < f(n'_old) )    // Compare cost of solution bases.
           THEN   // Solution base of n' is cheaper:  path discarding.
             n'_old.parent = n'.parent;  f(n'_old) = f(n');
             IF n'_old ∈ CLOSED THEN remove(n'_old, CLOSED); add(n'_old, OPEN, f(n'_old)); ENDIF
         ENDIF
```

State-space:



OPEN ∪ CLOSED list:

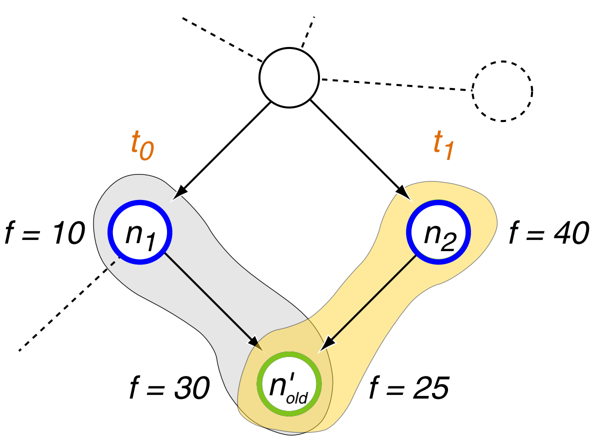


Re-evaluation of a Node n' (continued)

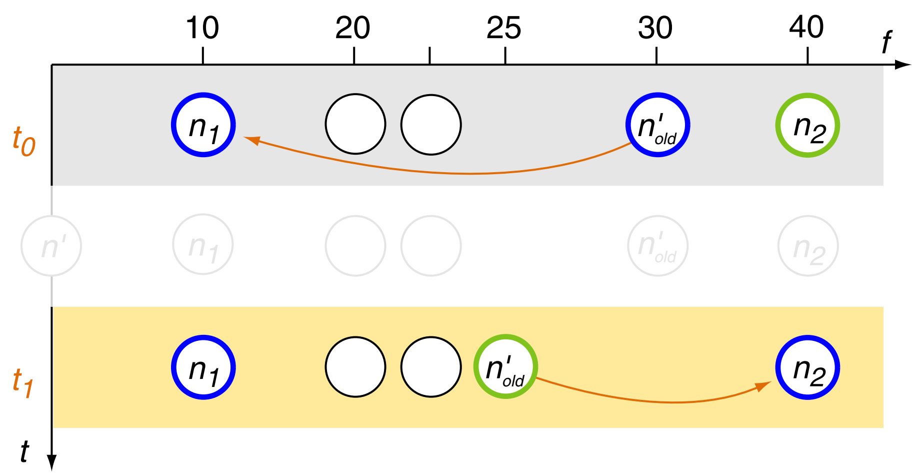
Case 2: n'_{old} is already on CLOSED.

```
5.   FOREACH n' IN successors(n) DO      // Expand n.
       ...
       n'_old = retrieve(n', OPEN ∪ CLOSED);    // State of n' already visited?
       IF ( n'_old == null )
       ...
       ELSE
         IF ( f(n') < f(n'_old) )    // Compare cost of solution bases.
         THEN    // Solution base of n' is cheaper:  path discarding.
           n'_old.parent = n'.parent;  f(n'_old) = f(n');
           IF n'_old ∈ CLOSED THEN remove(n'_old, CLOSED);  add(n'_old, OPEN, f(n'_old)); ENDIF
       ENDIF
```

State-space:



OPEN ∪ CLOSED list:



Remarks:

- Given an occurrence of Case 2, it follows that f is not a monotonically increasing function in the solution base size (path length): $f(n') < f(n_2)$.
- Q. Given Case 2, and given the additional information that n_2 is a descendant of n'. What does this mean?
- Case 1 and Case 2 illustrate the path discarding behavior of algorithm BF, it follows that f is not a monotonically increasing function in the solution base size (path length): $f(n') < f(n_2)$.
- Implementation / efficiency issue: Instead of reopening a node n' (i.e., instead of moving n' from CLOSED to OPEN), a recursive update of the f-values and the back-pointers of its successors can be done. This is highly efficient but should only be done with care as it can easily lead to inconsistent traversal trees (wrong back-pointers).

  After reopening a node n', all the nodes n'' from which n' is reachable using only back-pointers are still available. Since the f-values stored with such nodes n'' are not updated, subsequent node expansions may use f-values not matching back-pointer paths. This can cause additional search efforts. Performing node expansion for nodes with invalid f-values can be avoided by using order-preserving functions f. Reopening nodes can be avoided by using monotonically increasing functions f (i.e., $f(n) \leq f(n')$ for successors n' of n).

Best-First Search Algorithms

Re-evaluation of a Node n' (continued)

Case 3: n'_{old} has been on OPEN but is not found on OPEN or CLOSED.

```
5.    FOREACH n' IN successors(n) DO      // Expand n.
        ...
        n'_old = retrieve(n', OPEN ∪ CLOSED);     // State of n' already visited?
        IF ( n'_old == null )
   ➜    THEN   // n' not in OPEN or CLOSED:  n' is a new state.
          add(n', OPEN, f(n'));
        ELSE
          ...
        ENDIF
```

Possible reasons:

1. There is no occurrence check. (State-space graph G is modeled as a tree.)
2. The occurrence check does not work properly. Note that state recognition can be a very hard (even undecidable) problem.
3. Explored parts of the state-space graph that seemed to be no longer required have been deleted by *cleanup_closed*.

Best-First Search Algorithms

Re-evaluation of a Node n' (continued)

Case 3: n'_{old} has been on OPEN but is not found on OPEN or CLOSED.

```
5.    FOREACH n' IN successors(n) DO    // Expand n.
        ...
        n'_old = retrieve(n', OPEN ∪ CLOSED);    // State of n' already visited?
        IF ( n'_old == null )
→       THEN   // n' not in OPEN or CLOSED:  n' is a new state.
          add(n', OPEN, f(n'));
        ELSE
          ...
        ENDIF
```

Possible reasons:

1. There is no occurrence check. (State-space graph G is modeled as a tree.)
2. The occurrence check does not work properly. Note that state recognition can be a very hard (even undecidable) problem.
3. Explored parts of the state-space graph that seemed to be no longer required have been deleted by *cleanup_closed*.

Remarks:

- Q. What is the effect of the occurrence check in Case 1 and Case 2?
- Q. Should each visited node be stored in order to recognize the fact that its associated problem is encountered again?
- Q. Does a missing occurrence check affect the correctness of Algorithm BF?
- The shown version of the Algorithm BF has no call to *cleanup_closed*. However, such a call can be easily integrated, similar to the algorithms DFS or BFS.

Best-First Search Algorithms

$\text{BF}^*(s, \textit{successors}, \star, f)$ // A delayed termination variant of BF.

1. $s.\textit{parent} = \textit{null}$; $\textit{add}(s, \text{OPEN}, f(s))$;   // Store s on f-sorted OPEN.

2. **LOOP**

3. IF $(\text{OPEN} == \emptyset)$ THEN RETURN(*Fail*);

4. $n = \textit{min}(\text{OPEN}, f)$;   // Find most promising (cheapest) solution base.

➜ IF $\star(n)$ THEN RETURN(n);   // Delayed termination.

$\textit{remove}(n, \text{OPEN})$; $\textit{add}(n, \text{CLOSED})$;

5. **FOREACH** n' IN $\textit{successors}(n)$ **DO**   // Expand n.

$n'.\textit{parent} = n$;

➜ ~~IF $\star(n')$ THEN RETURN(n');~~   // Early termination removed.

$n'_{old} = \textit{retrieve}(n', \text{OPEN} \cup \text{CLOSED})$;   // State of n' already visited?

IF ($n'_{old} == \textit{null}$)

THEN

$\textit{add}(n', \text{OPEN}, f(n'))$;

ELSE

IF ($f(n') < f(n'_{old})$)

THEN   // Solution base of n' is cheaper: path discarding.

$n'_{old}.\textit{parent} = n'.\textit{parent}$; $f(n'_{old}) = f(n')$;

IF $n'_{old} \in \text{CLOSED}$ THEN $\textit{remove}(n'_{old}, \text{CLOSED})$; $\textit{add}(n'_{old}, \text{OPEN}, f(n'_{old}))$; ENDIF

ENDIF

ENDIF

ENDDO

6. **ENDLOOP**

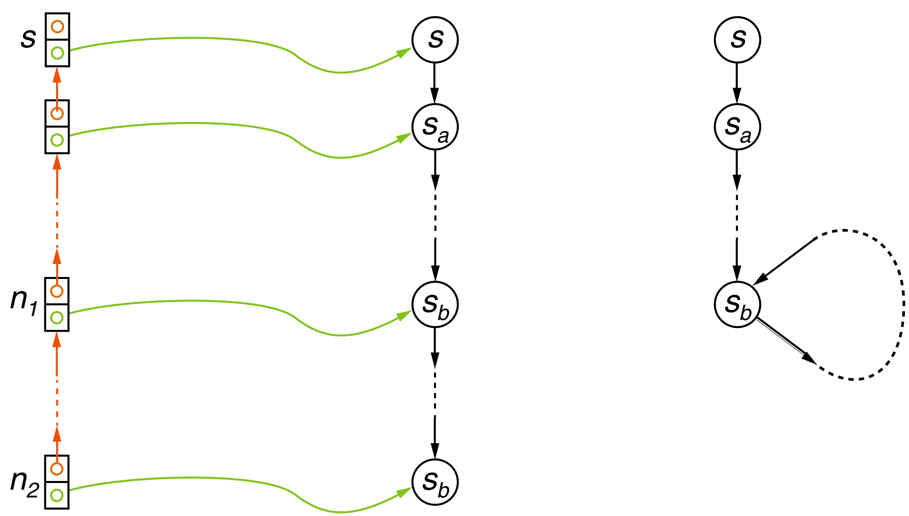
Best-First Search Algorithms

Definition 21 (Cycle-Averse Evaluation Function)

Let f be an evaluation function defined for state-space graph G.

f is called *cycle-averse*, if for each node n_2 with a cyclic back-pointer path, i.e., containing another node n_1 referring to the same state (n_1 is first occurrence, nearer to the start node s, and n_2 is some later occurrence), such that n_1 is reachable from n_2 via back-pointers, we have

$$f(n_1) \leq f(n_2) \qquad \text{i.e.,} \qquad f_{P_{s-n_1}}(n_1) \leq f_{P_{s-n_1-n2}}(n_2)$$



Back-pointer Structure   Traversal Tree   Path in G

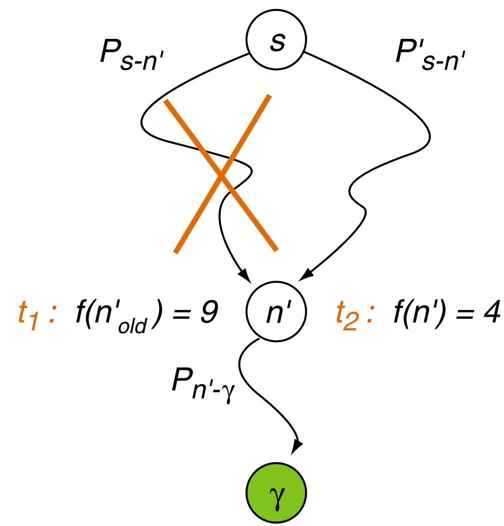
Remarks:

- If the task is to find a cheapest solution path that satisfies some constraints, we might not be successful when f is cycle-averse, even if path from start to goal nodes exist.

  As an example we can consider a minimum-path-length constraint, i.e., a solution path is required to have at least a path length of B for some B in $\mathbf{N}$. If a solution path exists, it might be necessary to "blow up" the path by adding cycles in order to meet the length constraint.

Best-First Search Algorithms

Irrevocable Path Discarding in BF



Path discarding is based on f-values computed for node instances.

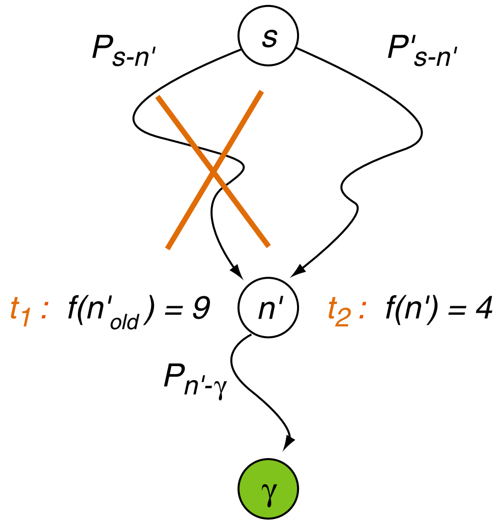
Irrevocability may not be allowable (solutions missed) if constraints on solution paths take into account *global properties* of the path.

Examples:

1. "Determine the shortest path (cheapest solution) that has two edges (operators) of equal costs."
2. "Determine a path (a solution) that minimizes the maximum edge cost difference (operator cost difference)."

Best-First Search Algorithms

Irrevocable Path Discarding in BF (continued)



Irrevocability is reasonable:

1. For constraint satisfaction problems, if the following equivalence holds:

   "Solution base $P_{s-n'}$ can be completed by $P_{n'-\gamma}$ to a solution path."

   $\Leftrightarrow$

   "Solution base $P'_{s-n'}$ can be completed by $P_{n'-\gamma}$ to a solution path."

2. For optimization problems, if for alternative solution bases the order w.r.t. cost estimations is preserved when using $P_{n'-\gamma}$ as their shared continuation.

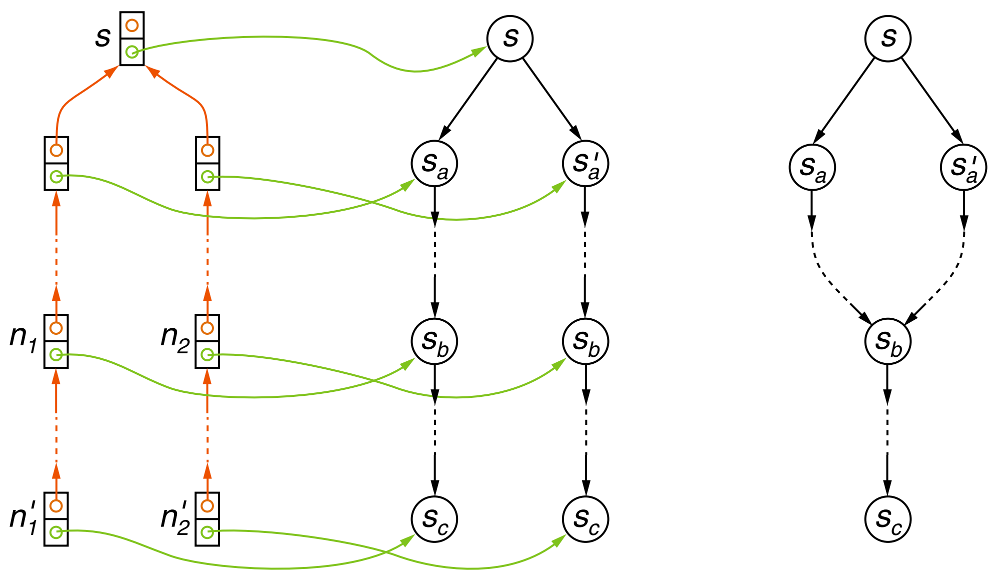
Best-First Search Algorithms

Definition 22 (Order-preserving Evaluation Function)

Let f be an evaluation function defined for state-space graph G.

f is called *order-preserving*, if for each pair of nodes n_1' and n_2' with predecessors n_1 and n_2 via back-pointers respectively, such that the back-pointer paths of n_1' and n_2' coincide from n_1 resp. n_2 on, then we have

$$f(n_1) \leq f(n_2) \ \Rightarrow f(n_1') \leq f(n_2')$$



Back-pointer Structure   Traversal Tree   Paths in G

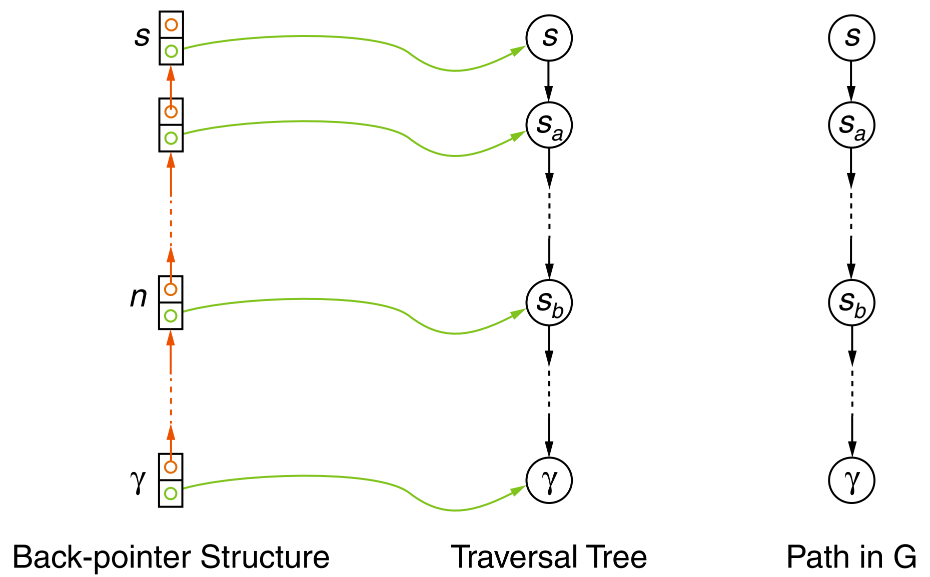
Best-First Search Algorithms

Definition 23 (Optimistic Evaluation Function)

Let G be state-space graph and f an evaluation function for G.

f is called *optimistic*, if for each goal node γ and each predecessor node n in the back-pointer path of γ (n reachable from γ via back-pointers), we have

$$f(n) \leq f(\gamma)$$



Remarks:

- Let G be a state-space graph with non-negative cost values assigned to the edges. Let the evaluation function f be defined by

  $$f_{P_{s_0-s_1}}(s_1) = \text{sum of edge cost value in } P_{s_0-s_1}.$$

  Then f is optimistic.

Best-First Search Algorithms

Advanced Principles for an Algorithmization of Best-First Search for Optimization

Prop$_{BF}(G)$ Required Properties of G for Optimization

1. G has *Prop*$_1(G)$ properties.
2. f is cycle-avers. (Avoiding corrupted backpointer structures.)
3. f is order-preserving. (Avoiding path discarding problems.)

Additional property (kept separate as usual):

- f is optimistic. (Avoiding overestimation problems.)

Task

- Determine an optimum solution path for s in G.

Algorithmization

- The algorithm uses *Delayed Termination*. (Avoiding last step problems.)
- The algorithm uses *Path Discarding*. (Efficiency.)
- The tie breaking strategy for OPEN prefers goal nodes.

Definition 24 (Admissibility)

Let $\mathcal{A}$ be an algorithm searching a state-space graph G for a solution path for a given state s.

$\mathcal{A}$ is *admissible* if
$\mathcal{A}$ terminates returning an optimum (with respect to f) solution if a solution exists.

There is no guarantee for the existence of an optimum solution path, even if a solution path exists.

State Space Search

Lemma 25 (Admissibility of BF* for Finite Graphs)

Let G be for finite graphs G with *Prop*$_{BF}(G)$ and let f be an optimistic evaluation function for G. Then BF* is admissible.

Proof (sketch)

1. Since G is finite, the number of cycle-free solution paths starting in s is finite. Hence, a minimum cost solution path $P_{s-\gamma}$ exists in G. (Only cycle-free solution paths have to be considered, since f is cycle-averse and order-preserving.)

2. Assume, BF* terminates returning a non-optimum solution $P_{s-\gamma'}$.
   Hence, $f(\gamma) < f(\gamma')$.

3. At each point in time (whenever BF* is in step 2) before BF* terminates, there is a shallowest node n in $P_{s-\gamma}$ that is in OPEN.
   (Shallowest node in a path is the node nearest to the start node.)
   Hence, BF* cannot terminate with *Fail*.

4. A shallowest OPEN node on an optimum path is optimally reached, i.e., there is no path from s to n with a smaller f-value than that the current back-pointer path.

5. Since f is optimistic, we have $f(n) \leq f(\gamma)$.

6. This contradicts the termination returning $P_{s-\gamma'}$, since goal node γ' was selected from OPEN when also n was available on OPEN.