Chapter S:III

III. Informed Search

- ❑ Best-First Search Basics
- ❑ Best-First Search Algorithms
- ❑ Cost Functions for State-Space Graphs
- ❑ Evaluation of State-Space Graphs
- ❑ Algorithm A*

- ❑ BF* Variants
- ❑ Hybrid Strategies

- ❑ Best-First Search for AND-OR Graphs
- ❑ Relation between GBF and BF
- ❑ Cost Functions for AND-OR Graphs
- ❑ Evaluation of AND-OR Graphs

Best-First Search for AND-OR Graphs

Principles of Algorithm GBF [BF principles]

Properties of Problem-Reduction Graphs

1. G is an implicitly defined directed AND-OR graph.
2. G is acyclic and locally finite.

When searching problem-reduction graphs with the Algorithm GBF,

- ❑ the underlying problem-reduction graph G is assumed to be acyclic,
- ❑ f_1 evaluates solution bases H,
- ❑ a most promising solution base minimizes f_1,
- ❑ a most promising solution base is searched *among all solution bases* maintained by GBF,
- ❑ solution bases maintained by GBF are solution bases for s in the explicit part G_e of G with tip nodes in OPEN,
- ❑ f_2 returns a most informative node in a solution base.

Best-First Search for AND-OR Graphs

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- a most promising solution base is searched *among all solution bases* maintained by GBF,
- solution bases maintained by GBF are solution bases for s in the explicit part G_e of G with tip nodes in OPEN,
- f_2 returns a most informative node in a solution base.

Remarks:

- Again, G denotes the explored part of the search space graph.
- When searching AND-OR graphs with the algorithm GBF, only solution bases H need to be considered that have not been considered so far. I.e., only solution bases H with leaf nodes that are goal nodes or nodes in `OPEN` and inner nodes in `CLOSED` are considered.
- In an AND-OR graph, a node may be part of several solution bases.
- Usually, the evaluation function $f_1(H)$ is based on a heuristic $h(n)$.
- $h(n)$ estimates the optimum cost of solution graphs for the rest problem associated with a node n. Ideally, $h(n)$ should consider the probability of the solvability of the problem at node n.
- Compared to $f_2(H)$, the evaluation function $f_1(H)$ is usually more important.
- Searching AND-OR graphs with cycles is intricate. Note that the algorithms presented by Martelli and Montanari [1973, 1978], Nilsson [1980], or Pearl [1984] presume graphs without cycles. In the following, we will restrict to acyclic graphs as well.
- In the context of Markov Decision Processes, MDP, and advanced planning algorithms researchers have dealt with AND-OR graphs that contain cycles. [Bonet/Geffner 2005]

Best-First Search for AND-OR Graphs

Algorithm:	GBF
Input:	s. Start node representing the initial problem. *successors*(n). Returns the successors of node n. $f_1(H)$. Evaluation function for solution bases. $f_2(H)$. Selection function for OPEN-nodes in a solution base.
Output:	A solution graph or the symbol *Fail*.
Subroutines:	*is_solved*(n). Predicate that is *True* if n is labeled "solved" or n is a goal node. *propagate_label*(n). Function that propagates node label "solved" along backpointers.

Best-First Search for AND-OR Graphs

```
GBF($s$, $\mathit{successors}$, $\mathit{is\_solved}$, $f_1$, $f_2$)

  $\mathit{insert}(s, \text{OPEN})$;  $\mathit{add\_node}(s, G_e)$;    // $G_e$ is the explored part of $G$.
  LOOP
    IF (OPEN $= \emptyset$) THEN RETURN($\mathit{Fail}$);



  ENDLOOP
```

Best-First Search for AND-OR Graphs

```
GBF($s$, $successors$, $is\_solved$, $f_1$, $f_2$)

    $insert(s, \texttt{OPEN})$;  $add\_node(s, G_e)$;    // $G_e$ is the explored part of $G$.
    LOOP
        IF ($\texttt{OPEN} = \emptyset$) THEN RETURN($Fail$);
        $H = min\_solution\_base(s, G_e, f_1)$;    // Find most prom. sol. base in $G_e$.
        $n = min(\texttt{OPEN} \cap H, f_2)$;    // Find most informative non-goal node in $H$.
        $remove(n, \texttt{OPEN})$;  $push(n, \texttt{CLOSED})$;

    ENDLOOP
```

Best-First Search for AND-OR Graphs

```
GBF($s$, successors, is_solved, $f_1$, $f_2$)

 1.  insert($s$, OPEN);  add_node($s$, $G_e$);    // $G_e$ is the explored part of $G$.
 2.  LOOP
 3.    IF (OPEN $= \emptyset$) THEN RETURN(Fail);
4.a    $H$ = min_solution_base($s$, $G_e$, $f_1$);    // Find most prom. sol. base in $G_e$.
4.b    $n$ = min(OPEN $\cap$ $H$, $f_2$);    // Find most informative non-goal node in $H$.
       remove($n$, OPEN);  push($n$, CLOSED);
 5.    FOREACH $n'$ IN successors($n$) DO



           insert($n'$, OPEN);  add_node($n'$, $G_e$);

         add_backpointer($n'$, $n$);  add_edge(($n$, $n'$), $G_e$);
         IF is_solved($n'$) // Is $n'$ goal node                    ?
         THEN
           propagate_label($n'$);
           IF is_solved($s$) THEN RETURN(compute_solution_graph($G_e$));
         ENDIF
       ENDDO
 6.  ENDLOOP
```

Best-First Search for AND-OR Graphs [Basic_AND-OR, BF, BF*, GBF*]

```
GBF($s$, successors, is_solved, $f_1$, $f_2$)

insert($s$, OPEN); add_node($s$, $G_e$);   // $G_e$ is the explored part of $G$.
LOOP
  IF (OPEN $= \emptyset$) THEN RETURN(Fail);
  $H$ = min_solution_base($s$, $G_e$, $f_1$);   // Find most prom. sol. base in $G_e$.
  $n$ = min(OPEN $\cap H$, $f_2$);   // Find most informative non-goal node in $H$.
  remove($n$, OPEN); push($n$, CLOSED);
  FOREACH $n'$ IN successors($n$) DO

    IF ( $n' \in$ OPEN OR $n' \in$ CLOSED )   // Instance of $n'$ seen before?
    THEN   // Use old instance of $n'$ instead.
      $n'$ = retrieve($n'$, OPEN $\cup$ CLOSED);
    ELSE   // $n'$ encodes an instance of a new state.
      insert($n'$, OPEN); add_node($n'$, $G_e$);
    ENDIF
    add_backpointer($n'$, $n$); add_edge(($n$, $n'$), $G_e$);

    IF is_solved($n'$) // Is $n'$ goal node or labeled solvable?
    THEN
      propagate_label($n'$);
      IF is_solved($s$) THEN RETURN(compute_solution_graph($G_e$));
    ENDIF

  ENDDO
ENDLOOP
```

Remarks:

- Finding dead ends can be handled in the same way as finding goal nodes.

```
IF ( $\perp(n')$ OR is_labeled$(n')$ )    // Is $n'$ unsolvable?
THEN
   propagate_label$(n')$; IF is_unsolvable$(s)$ THEN RETURN$(Fail)$;
ENDIF
```

- The expansion of nodes in G can be implemented with or without occurrence check:

  (a) With occurrence check. If an already existing rest problem is encountered, only an additional link is added to G_e. As a consequence, G_e is a subgraph of the underlying search space graph G.

  (b) Without occurrence check. Each successor becomes a new node, irrespective of duplicate rest problems. As a consequence, G_e corresponds to an AND-OR tree that can be considered as an unfolding of the underlying search space graph G.

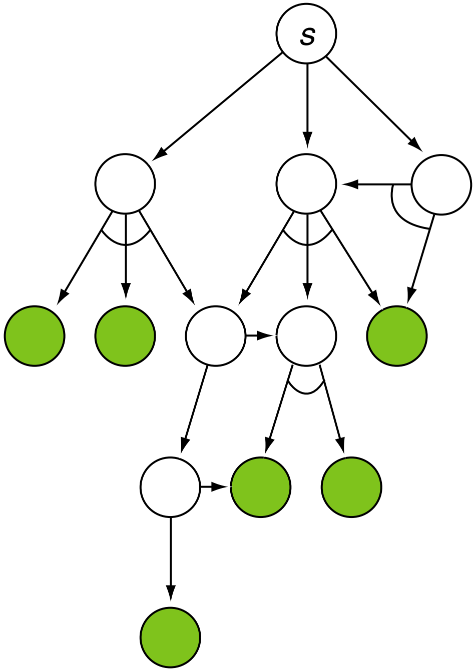
  Therefore, using an occurrence check or not can be seen as different ways of modeling a problem.

- The construction of solution bases depends strongly on the structure of the evaluation function f_1. In the general case, an individual construction of all possible solution bases plus an individual computation of the f_1-values cannot be avoided. However, if f_1 can be stated in a recursive manner using specific functions, computation of the minimum f_1 value and computation of a most promising solution base can be done very efficiently.

- General Best-First Algorithms are informed versions of Basic_AND-OR_Search.

Best-First Search for AND-OR Graphs

Illustration of GBF



○ Node on OPEN

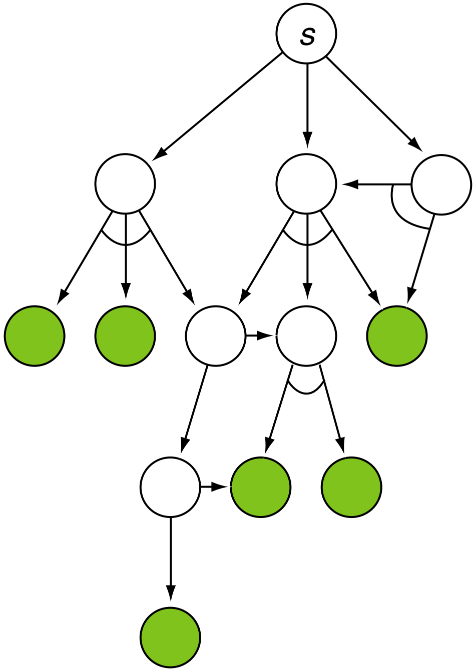
○ Node on CLOSED

● Solved rest problem

The underlying AND-OR graph of the following GBF illustration.

Best-First Search for AND-OR Graphs

Illustration of GBF



○ Node on OPEN

○ Node on CLOSED

● Solved rest problem

$f_1(H)$ Evaluates solution bases H in regard to their edge count.
Answers the question: What does the solution associated with H cost?

$f_2(H)$ Returns the node with a most expensive rest problem in H.
Answers the question: What node to expand next?

$h(n)$ Estimates the number of edges for the rest problem at node n.

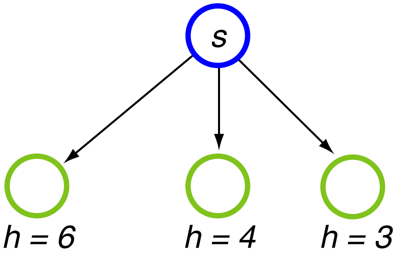
Remarks:

- The evaluation function $f_1(H)$ estimates the total number of edges of a solution graph extending solution base H. Obviously, $f_1(H)$ will be based on the heuristic $h(n)$.
- Rationale for f_1. Searching the smallest solution may quickly lead to a first solution graph. Keyword: *Small-is-Quick Principle*
- Rationale for f_2. A possible estimation error of h, which allegedly claims an unsolvable rest problem as a solvable one, should be detected earliest. Since it is reasonable to assume that the estimation error is proportional to the cost estimate, we pick a most expensive rest problem first.
- Recall that the complete AND-OR graph is neither given nor manageable for real-world problems.

Best-First Search for AND-OR Graphs

Illustration of GBF (continued)

1. s is expanded and moved to CLOSED.



○ Node on OPEN

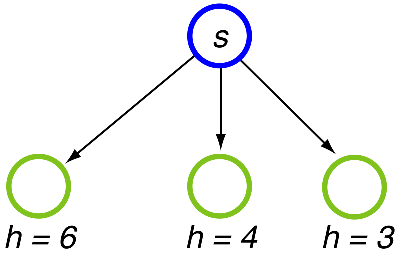
○ Node on CLOSED

● Solved rest problem

Best-First Search for AND-OR Graphs

Illustration of GBF (continued)

1. s is expanded and moved to CLOSED.

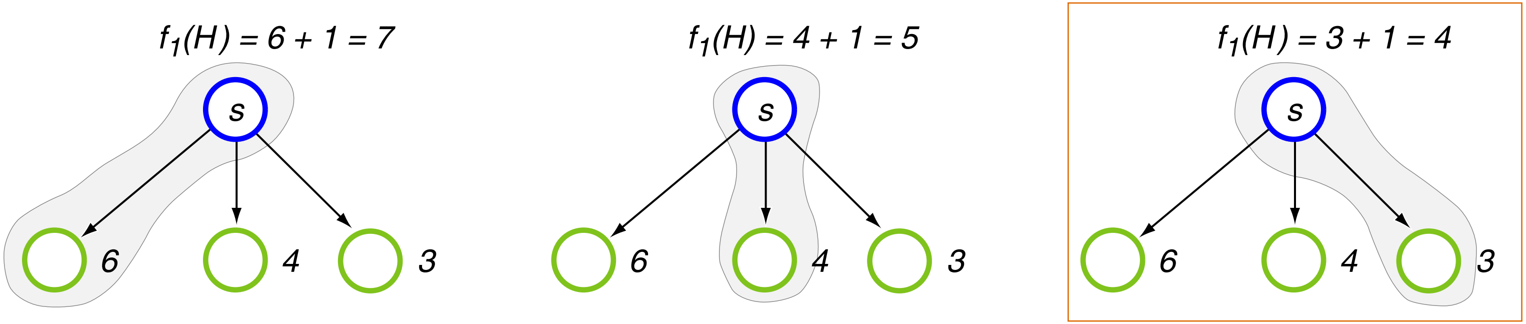


○ Node on OPEN

○ Node on CLOSED

● Solved rest problem

Possible solution bases:

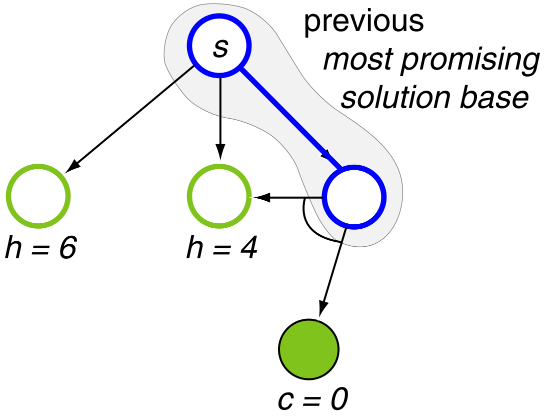


Most promising solution base H

Best-First Search for AND-OR Graphs

Illustration of GBF (continued)

2. Expansion of the node $n \in (\text{OPEN} \cap H)$.



○ Node on OPEN

○ Node on CLOSED

● Solved rest problem

Best-First Search for AND-OR Graphs

Illustration of GBF (continued)

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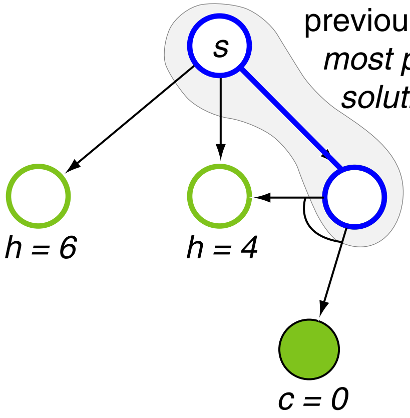
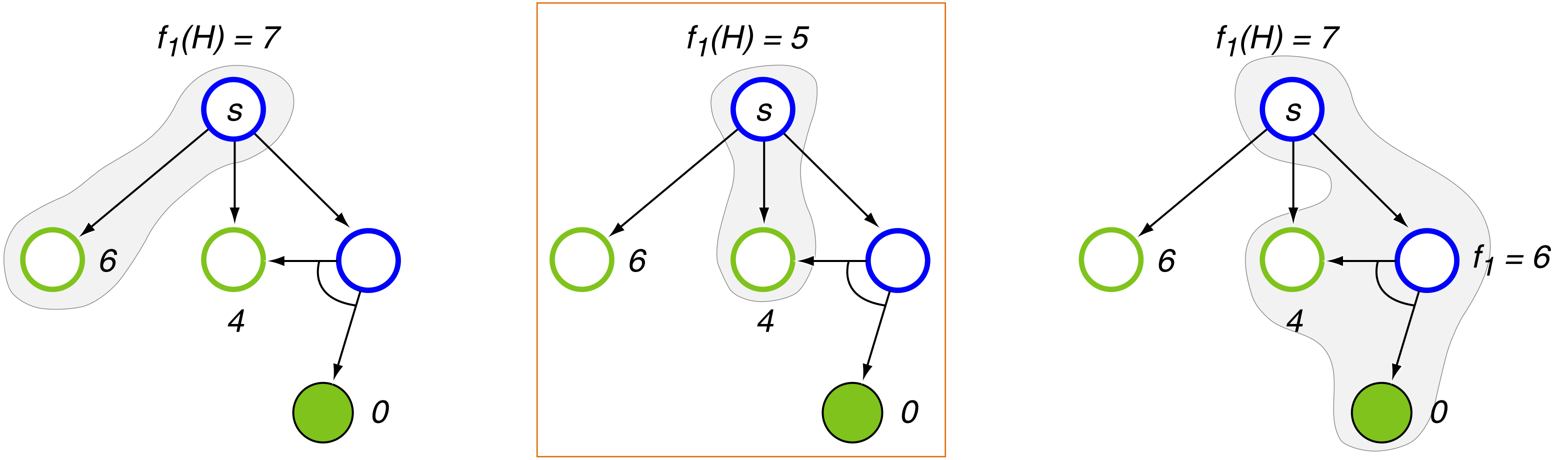


previous *most promising solution base*

- ○ Node on OPEN
- ○ Node on CLOSED
- ● Solved rest problem

Possible solution bases:

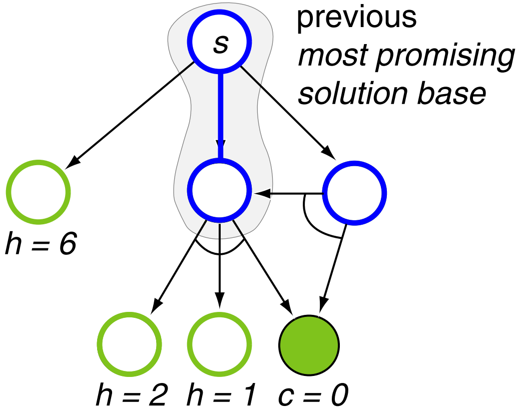


Most promising solution base H

Best-First Search for AND-OR Graphs

Illustration of GBF (continued)

3. Expansion of the node $n \in (\text{OPEN} \cap H)$.



○ Node on OPEN

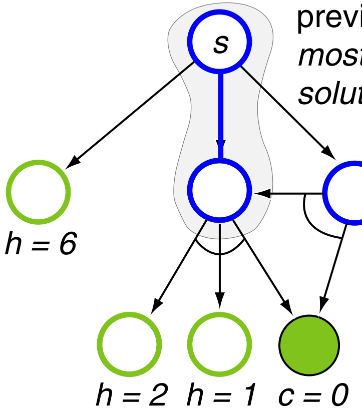
○ Node on CLOSED

● Solved rest problem

Best-First Search for AND-OR Graphs

Illustration of GBF (continued)

3. Expansion of the node $n \in (\text{OPEN} \cap H)$.



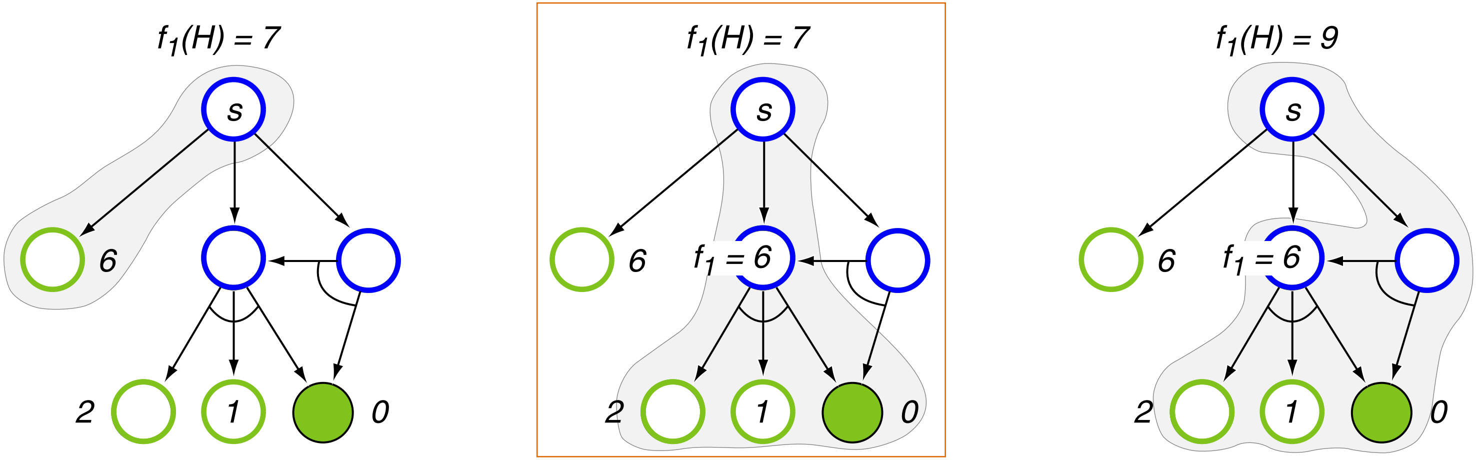
previous
most promising
solution base

○ Node on OPEN

○ Node on CLOSED

● Solved rest problem

Possible solution bases:



$f_1(H) = 7$

$f_1(H) = 7$

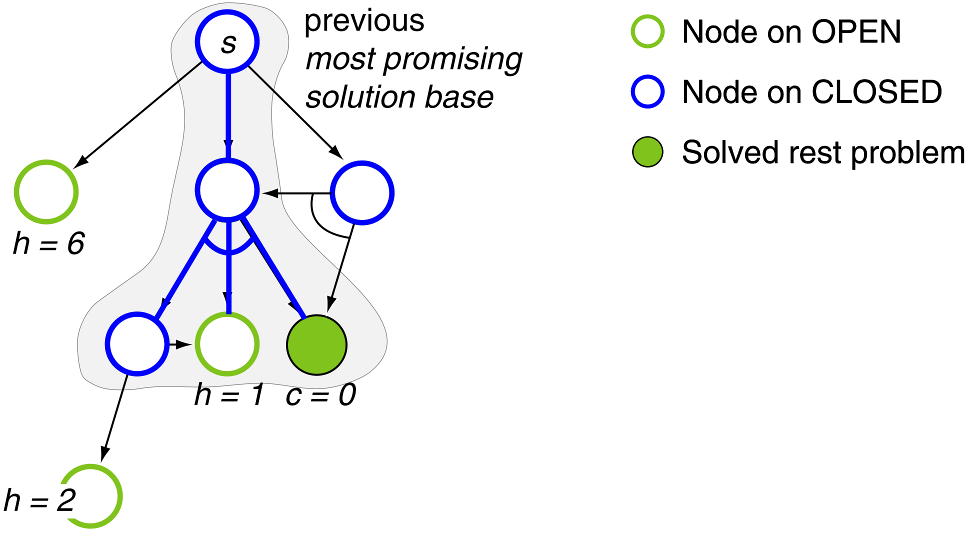
$f_1(H) = 9$

Most promising solution base H

Best-First Search for AND-OR Graphs

Illustration of GBF (continued)

4. Based on $f_2(n)$: expansion of some node $n \in (\text{OPEN} \cap H)$.



Best-First Search for AND-OR Graphs

Illustration of GBF (continued)

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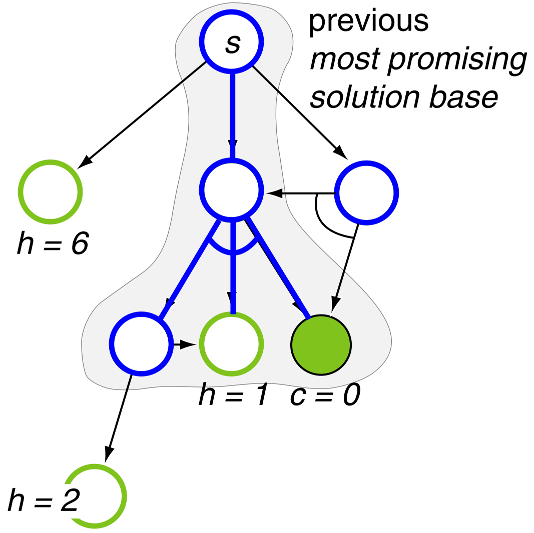


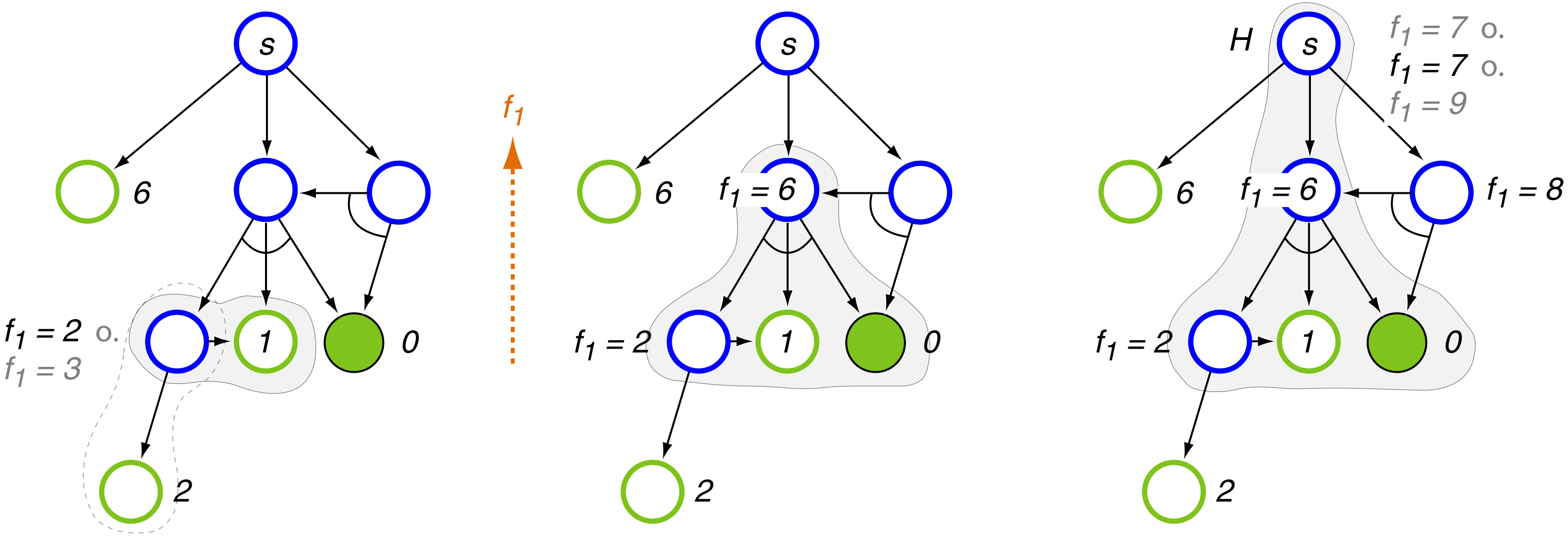
previous *most promising solution base*

○ Node on OPEN

○ Node on CLOSED

● Solved rest problem

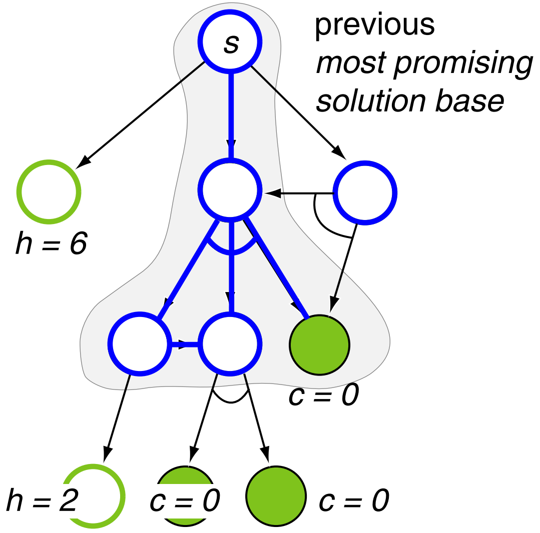
Solution bases:



Best-First Search for AND-OR Graphs

Illustration of GBF (continued) [next step]

5. Expansion of the node $n \in (\text{OPEN} \cap H)$.



○ Node on OPEN

○ Node on CLOSED

● Solved rest problem

Best-First Search for AND-OR Graphs

Illustration of GBF (continued) [next step]

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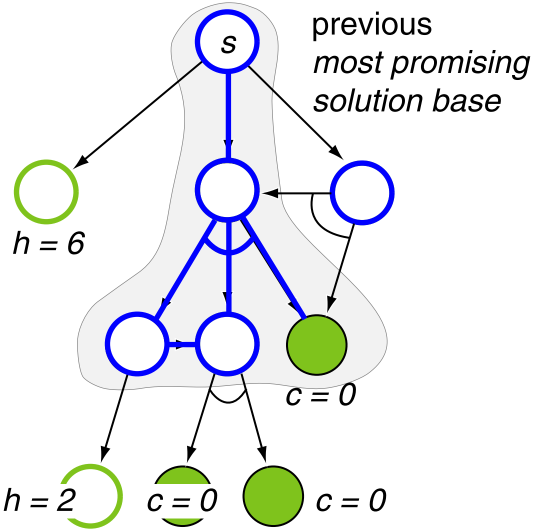
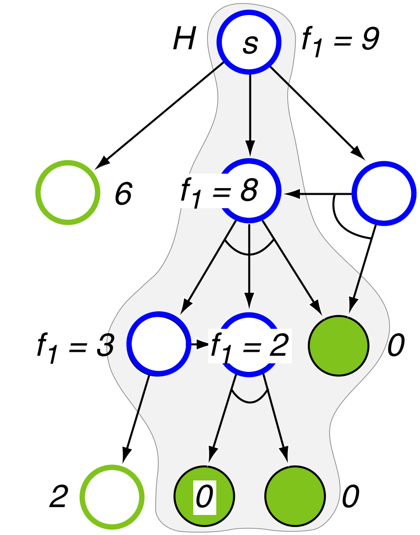


○ Node on OPEN

○ Node on CLOSED

● Solved rest problem

A solution graph H:



Best-First Search for AND-OR Graphs

Illustration of GBF (continued) [next step]

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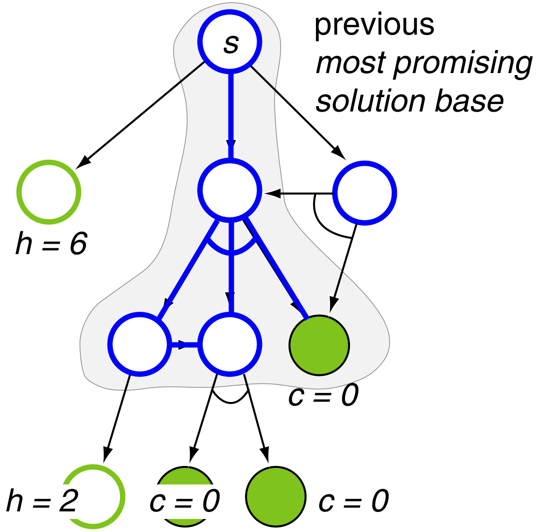

○ Node on OPEN

○ Node on CLOSED

● Solved rest problem

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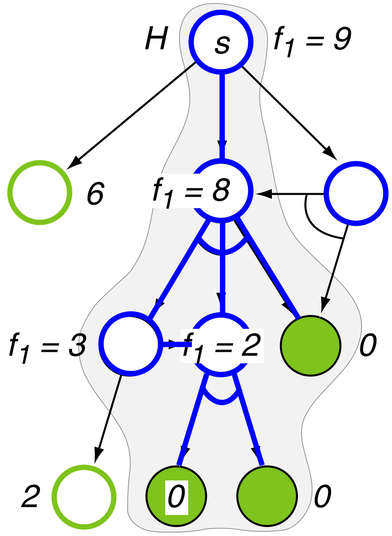
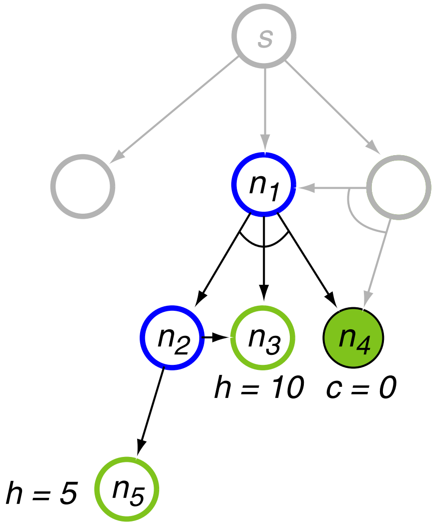


Best-First Search for AND-OR Graphs

Multiple Accounting of a Node



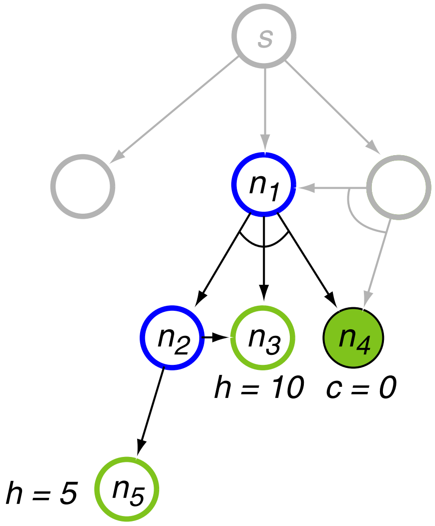
○ Node on OPEN

○ Node on CLOSED

● Solved rest problem

Best-First Search for AND-OR Graphs

Multiple Accounting of a Node



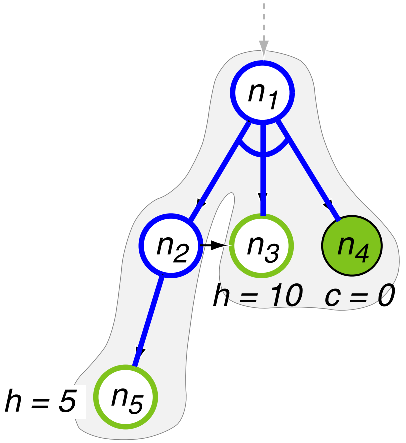
○ Node on OPEN

○ Node on CLOSED

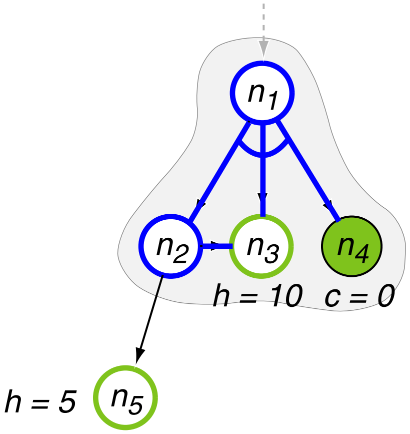
● Solved rest problem

n_1 start node

(a)



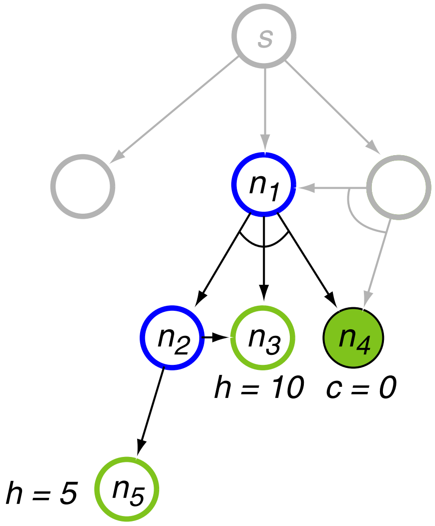
(b)



$C_G(n_1)$	(a)	(b)
duplicate count		
true count		

Best-First Search for AND-OR Graphs

Multiple Accounting of a Node

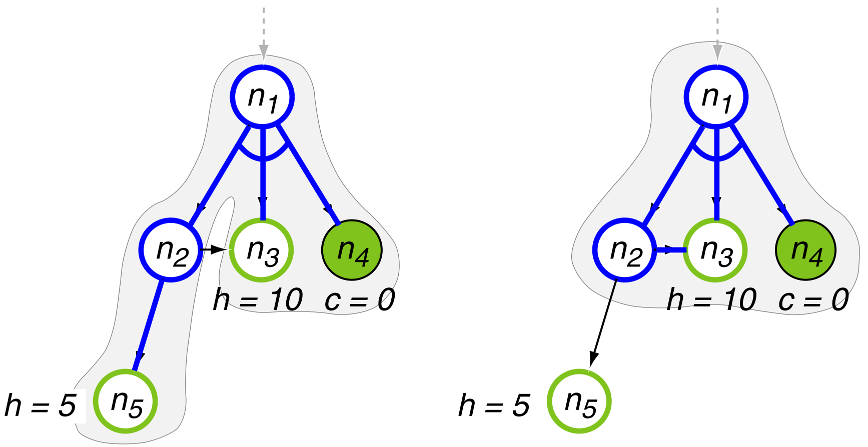


○ Node on OPEN

○ Node on CLOSED

● Solved rest problem

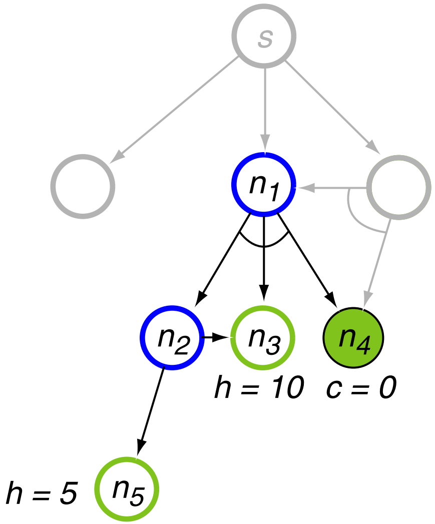
n_1 start node (a) (b)



$C_G(n_1)$	(a)	(b)
duplicate count	**19**	24
true count		

Best-First Search for AND-OR Graphs

Multiple Accounting of a Node



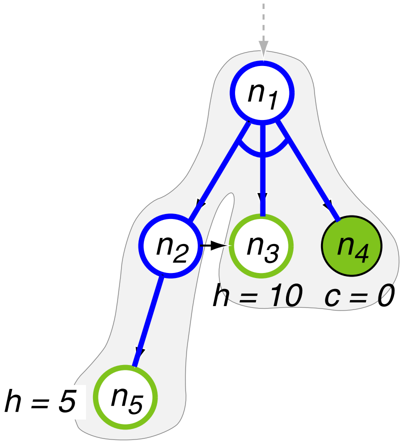
○ Node on OPEN

○ Node on CLOSED

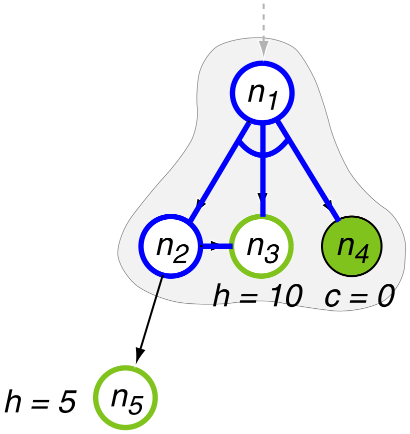
● Solved rest problem

n_1 start node

(a)



(b)



$C_G(n_1)$	(a)	(b)
duplicate count	**19**	24
true count	19	**14**

Remarks:

- The minimum cost contribution of node n_2 to a solution is not constant, depending on the multiple accounting of node n_3.
- Q. Which approach is to be preferred—simple computation of f_1 ($\sim$ multiple accounting) or true accounting?

Best-First Search for AND-OR Graphs

Optimality of Algorithm GBF [BF optimality]

In general, the first solution found by Algorithm GBF may not be optimum with respect to the evaluation function f_1.

Important preconditions for (provably) finding optimum solution graphs in AND-OR-graphs by general best-first algorithms:

1. The cost estimate underlying f_1 must be optimistic, i.e., underestimating costs or overestimating merits.

   In particular, the true cost of a cheapest solution graph H extending a solution base H exceeds its f_1-value: $C_H(s) \geq f_1(H)$.

2. The termination in case of success (*is_solved*$(s) =$ *True*) must be delayed.

   In particular, there is no call to *solved_labeling* after node expansion, but each time when determining a most promising solution base H.

   Algorithm GBF with delayed termination is called Algorithm GBF*.

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Best-First Search for AND-OR Graphs [BF, BF$\star$, GBF]

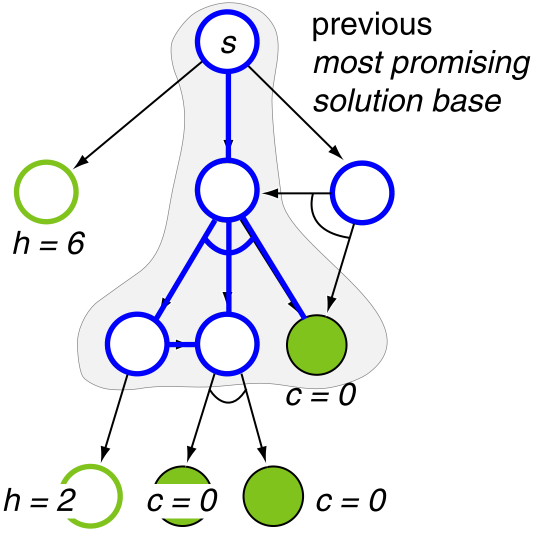
GBF*(s, *successors*, *is_solved*, f_1, f_2)

```
 1.  insert($s$, OPEN);  add_node($s$, $G_e$);    // $G_e$ is the explored part of $G$.
 2.  LOOP
 3.    IF (OPEN $= \emptyset$) THEN RETURN(Fail);
4.a    $H$ = min_solution_base($s$, $G_e$, $f_1$);    // Find most prom. sol. base in $G_e$.
 ➜     solved_labeling($H$);    // Is $H$ a solution graph?
 ➜     IF is_solved($s$) THEN RETURN($H$);    // Delayed termination.
4.b    $n$ = min(OPEN $\cap$ $H$, $f_2$);    // Find most informative non-goal node in $H$.
       remove($n$, OPEN);  push($n$, CLOSED);
 5.    FOREACH $n'$ IN successors($n$) DO
         IF ( $n' \in$ OPEN OR $n' \in$ CLOSED )    // Instance of $n'$ seen before?
         THEN   // Use old instance of $n'$ instead.
           $n'$ = retrieve($n'$, OPEN $\cup$ CLOSED);
         ELSE   // $n'$ encodes an instance of a new state.
           insert($n'$, OPEN);  add_node($n'$, $G_e$);
         ENDIF
         add_backpointer($n'$, $n$);  add_edge(($n$, $n'$), $G_e$);

 ➜       ~~IF is_solved($n'$)~~
 ➜       ~~THEN~~
 ➜         ~~propagate_label($n'$); IF is_solved($s$) THEN RETURN(compute_solution_graph($G_e$));~~
 ➜       ~~ENDIF~~

       ENDDO
 6.  ENDLOOP
```

Best-First Search for AND-OR Graphs

Illustration of GBF* (continued) [previous step]

5. Expansion of the node $n \in (\text{OPEN} \cap H)$.



○ Node on OPEN

○ Node on CLOSED

● Solved rest problem

Best-First Search for AND-OR Graphs

Illustration of GBF* (continued) [previous step]

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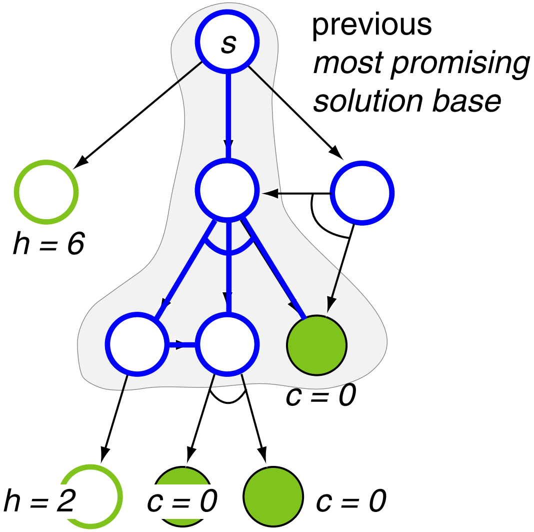
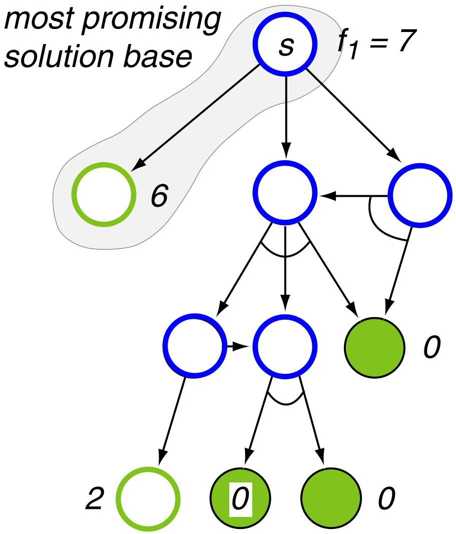


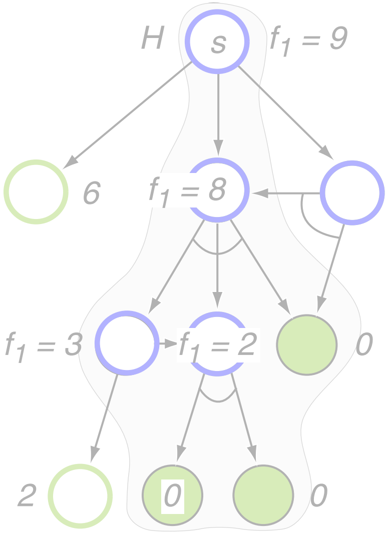
○ Node on OPEN

○ Node on CLOSED

● Solved rest problem

Solution base / graph:

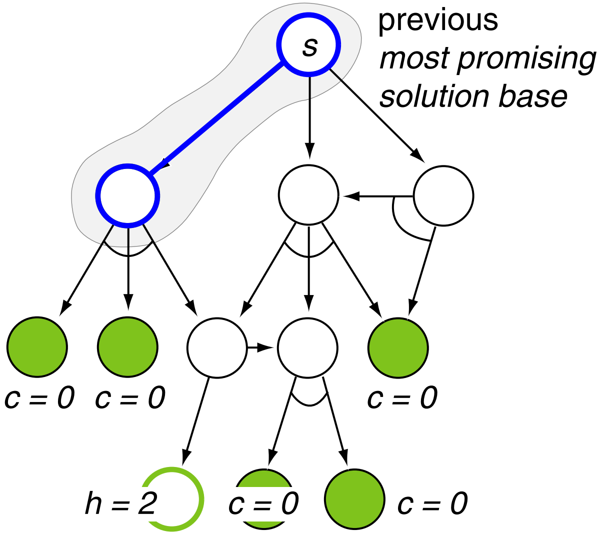




Best-First Search for AND-OR Graphs

Illustration of GBF* (continued)

6. Expansion of the node $n \in (\text{OPEN} \cap H)$.



○ Node on OPEN

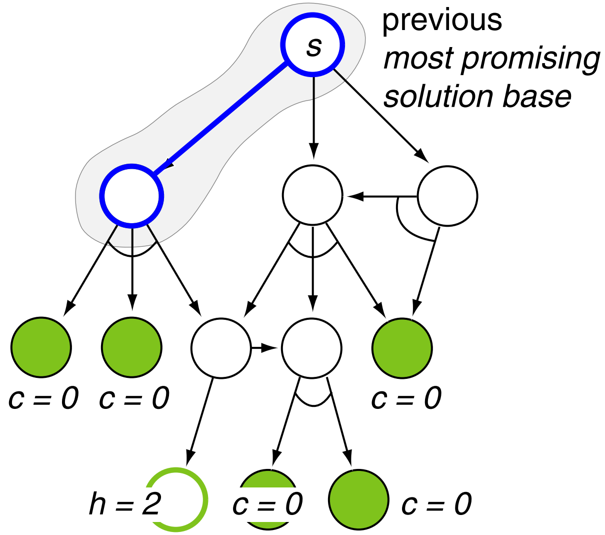
○ Node on CLOSED

● Solved rest problem

Best-First Search for AND-OR Graphs

Illustration of GBF* (continued)

6. Expansion of the node $n \in (\text{OPEN} \cap H)$.



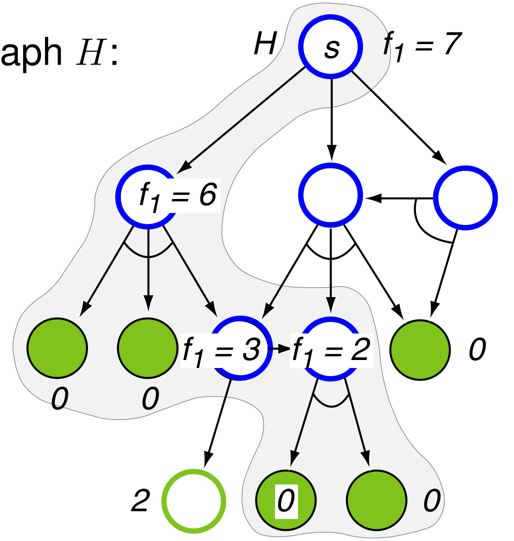
previous *most promising solution base*

○ Node on OPEN

○ Node on CLOSED

● Solved rest problem

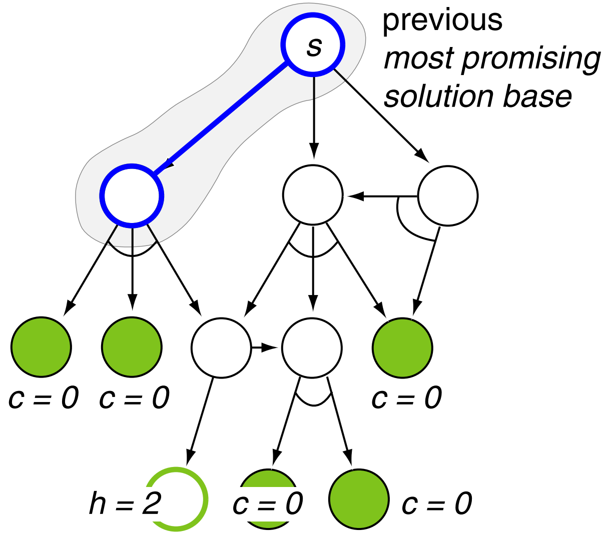
Returned solution graph H:



Best-First Search for AND-OR Graphs

Illustration of GBF* (continued)

6. Expansion of the node $n \in$ (OPEN $\cap$ H).

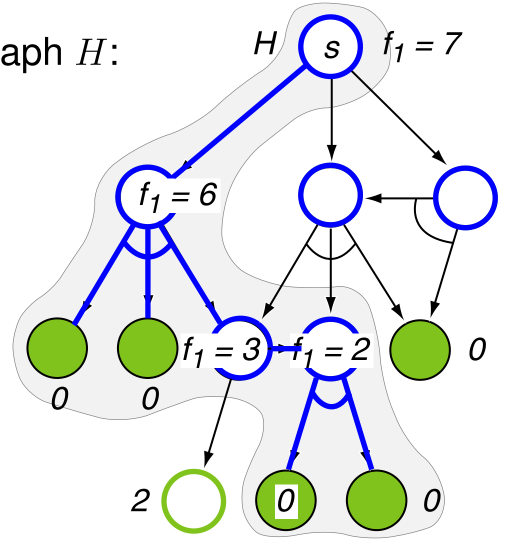


○ Node on OPEN

○ Node on CLOSED

● Solved rest problem

Returned solution graph H:



Relation between GBF and BF

Algorithm GBF applied to a state-space graph will simulate Algorithm BF.
(if the evaluation function f in BF is order preserving) [S:III Evaluation of State-Space Graphs]

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BF The discarding of a path (solution base) is *irrevocable*.

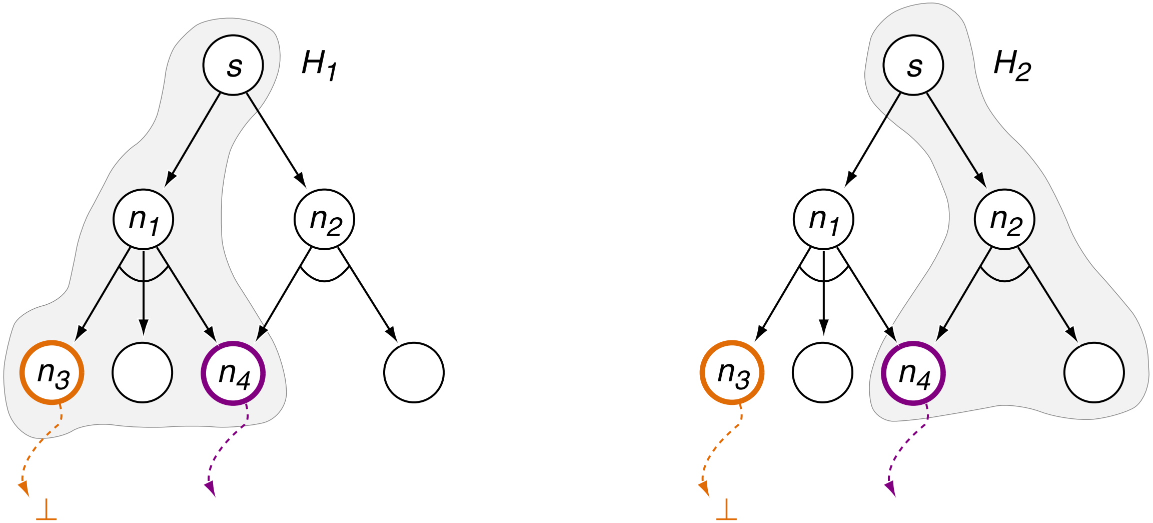
Remarks:

- (Under BF) the tree formed by the union of all solution bases is also called *traversal tree*. It is defined by the back-pointers that are set by the Algorithm BF.
- (Under BF) the decision to discard an inferior path (solution base) is based on a single parameter, usually a cost value, which is maintained at the respective OPEN node.
- Under BF the cost value assigned to a node is unique. Under GBF, however, a node can be used in different solution bases and may get assigned different cost values for different solution bases at the same time.
- Under BF there is a 1:1 relationship between expansion candidates (the nodes on OPEN) and solution bases (the paths that start at s and end at some node on OPEN). Put another way, each node on OPEN represents a solution base, and there is no solution base without a corresponding OPEN node.
- (Under BF) the term “irrevocable” means that an inferior path from s so some n will never become part of solution that also contains the node n. Irrevocability has wide-ranging consequences, which may be both crucial and reasonable, depending on the search problem at hand.

Relation between GBF and BF

Irrevocability is not Amenable to Problem Reduction Graphs

[Irrevocability in State-Space Graphs]



The following equivalence does not generally hold for problem reduction graphs:

$$
\begin{array}{rl}
 & \text{“Solution base } H_1 \text{ can be completed to a solution”} \\
\Leftrightarrow & \\
 & \text{“Solution base } H_2 \text{ can be completed to a solution.”}
\end{array}
$$

Remarks:

- The illustrated solution bases H_1, H_2, do not fulfill the equivalence. They share the continuation at node n_4, and let us assume that H_2 is inferior to H_1: $f_1(H_1) < f_1(H_2)$. However, H_2 cannot be discarded since at a later point in time the problem associated with node n_3 may turn out to be unsolvable. If we discarded H_2, GBF would miss the only possible solution.